International Academic Journal of Advanced Pure & Applied Science Research
ISSN: 5280-5269 | Volume 10, Issue 1 | January, 2025 | pages 47 – 56
OTL: 2229442831425272771015
Double Blind Peer Reviewed International Research Journal
https://arcnjournals.org
arcnjournalsa@gmail.com

Maximal Subsemigroups of the Semigroup of all Order-preserving and Order-decreasing Partial Injective Contractions of a Finite Chain

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Abstract: *Let $X_n = \{1,2, . . .,n\}$ be a finite chain and $ODCI_n$ be the semigroup of all order-preserving and order-decreasing partial injective contraction mappings. In this work the characterisation of the maximal subsemigroups of the semigroup $ODCI_n$ is obtained.*

Keywords: *Maximal, Finite Chain, Semigroup, Subsemigroup*

1 Introduction

Let $[n] = \{1,2,...,n\}$ be an n-elements set. A mapping α: $[n] \longrightarrow [n]$ is called transformation on $[n]$, the set of all transformations under the composition of mappings is called semigroup. Let $I_{[n]}$ denote the symmetric inverse semigroup the analogue of symmetric inverse group. A mapping whose domain is a subset of $[n]$ is called partial transfornation, if the domain is whole of $[n]$ is full transformation. The semigroup of all partial (full) transformations is denoted by $\mathcal{P}_{[n]}\left(\mathcal{T}_{[n]}\right)$ respectively, where as, $I_{[n]}$ consist of all partial one-one(injective) transformations. we denote by $dom(\alpha)$ and $im(\alpha)$ the domain and image of α respectively. A transformation $\alpha \in I_{[n]}$ is said to be orderpreserving(respectively order-reversing) if $x \le y \Longrightarrow x\alpha \le y\alpha$ (respectively $x\alpha \ge y\alpha$) for all $x,y \in dom(\alpha)$, is said to be order-decreasing(respectively, order-increasing) if for all $x \in dom(\alpha) \Longrightarrow x\alpha \le x$ (respectively, $x\alpha \ge x$), is isometry if $| x\alpha - y\alpha | = | x - y |$ and contraction or compression if $| x\alpha - y\alpha | \le | x - y |$. In this work, we will investigate the maximal subsemigroups and there number for the semigroup of all order-preserving and order-reversing partial injective contractions, $ODCI_{[n]}$.

We denote by $\mathcal{CP}_{[n]}$ $\left(\mathcal{CT}_{[n]}\right)$ and $CI_{[n]}$, the semigroup of partial(full) and partial one-one contractions respectively. The study of semigroups of contraction maps on a finite chain was initiated by Umar and Al-kharous [3], in which the notations for the new semigroups and its subsemigroups where proposed. Investigation on some semigroups of contractions were recorded, for details (see [1, 14, 10, 2, 15]). A subsemigroup M of a semigroup S is just a subset that is closed under the operation of S. A maximal subsemigroup M of a semigroup S is a proper subsemigroup that is not contained in any other proper subsemigroup, every proper subsemigroup of a finite semigroup is contained in a maximal subsemigroup[16].

There are alots of papers on finding the maximal subsemigroups of a particular classes of finite semigroups (for example; [6, 5, 13, 7, 9, 17, 8, 11]). Ganyushkin and Mazorchuk [4], described the maximal subsemigroups, maximal inverse subsemigroups and maximal nilpotent subsemigroups of the semigroup $IO_{[n]}$ the semigroup of all partial order-preserving injections. Dimitrova and Koppitz

[13] obtained the characterisation of the maximal subsemigrous of the ideals of $IO_{[n]}$. In Dimitrova [5] characterised the maximal subsemigroups of $ODP_{[n]}$ the semigroup of all partial order-preserving isometries a subsemigroup of all partial order-preserving transformations.

In the present work, we obtained the characterisation of the maximal subsemigroups of the semigroup $ODCI_{[n]}$. we will adopt standard notations and consider the use of some definitions in what follows, this paper is the analogue of Dimitrova [5].

An elament $\alpha \in ODCI_{[n]}$ can be written in the following tabular form

$$\alpha = \begin{pmatrix} a_1 & a_2 & . & . & . & a_r \\ b_1 & b_2 & . & . & . & b_r \end{pmatrix}, \qquad (1 \leq r \leq n) \qquad (1)$$

$b_i \leq a_i$ and $| x\alpha - y\alpha | \leq | x - y |$ for all $i \in [n]$. we will multiply mappings

from left to right and use the corresponding notation for the left to right compotion of transformations $x(\alpha\beta) = (x\alpha)\beta$, for all $\alpha,\beta \in ODCI_{[n]}$. Since α is injective $| dom(\alpha) | = | im(\alpha) |$ called the rank $rank(\alpha)$.

2 Ideals and starred Green's relations

Recall the characterisation of the ideals and starred Green's relation in [18]

Proposition 1 [[*18*],Theorem 1, 2 & 3] *Let $\alpha \in ODCI_{[n]}$, then the principal left(right) and two sided ideals generated by α has the forms*

(i) $\mathcal{ODCI}_{[n]}\alpha = \{\beta \in \mathcal{ODCI}_{[n]} : im(\beta) \subseteq im(\alpha)\}$

(ii) $\alpha\mathcal{ODCI}_{[n]} = \{\beta \in \mathcal{ODCI}_{[n]} : dom(\beta) \subseteq dom(\alpha)\}$

(iii) $\mathcal{ODCI}_{[n]}\alpha\mathcal{ODCI}_{[n]} = \{\beta \in \mathcal{ODCI}_{[n]} : rank(\beta) \leq rank(\alpha)\}$.

Proof 1 *(i) Let $\alpha \in ODCI_n$, suppose $A = \{\beta \in ODCI_n : im(\beta) \subseteq im(\alpha)\}$. If δ*

$\in ODCI_n$, we have $(x)\delta\alpha = (x\delta)\alpha$. by [[*18*],lemma 4] *$im(\delta\alpha) \subseteq im(\alpha)$ and thus $ODCI_n\alpha \subseteq A$. Conversely, consider an arbitiry $\beta \in A$. We have $im(\beta) \subseteq im(\alpha)$. For each $b \in im(\beta)$ choose some a_b such that $(a_b)\alpha = b$. Consider a transformation δ for which $dom(\delta) = dom(\beta)$ and such that for $x \in dom(\beta)$ we have $x\delta = a_b$ and thus $(x)\delta\alpha = ((x)\delta)\alpha = (a_b)\alpha = b$, implies that $A \subseteq ODCI_n\alpha$ as required.*

Proof 2 *(ii) Let $A = \{\beta \in ODCI_n : dom(\beta) \subseteq dom(\alpha)\}$. Suppose $\delta \in ODCI_n$ such that, $\alpha\delta = \beta$, we have $dom(\alpha\delta) = dom(\beta)$, but by* [[*18*],lemma 4] *$dom(\alpha\delta) \subseteq dom(\alpha) \subseteq dom(\beta)$, thus, $ODCI_n \subseteq A$.*

Conversely, suppose $\beta \in A$ and $dom(\beta) \subseteq dom(\alpha)$, implies that there exist δ in $ODCI_n$ such that $\beta = \alpha\delta$ but $\alpha\delta \in \alpha ODCI_n$. Hence $A \subseteq ODCI_n$.

Proof 3 *(iii) Let $D = \{\beta \in ODCI_n : rank(\beta) \leq rank(\alpha)\}$, by*[[*18*],lemma 4]

(iii) we have $ODCI_n\alpha ODCI_n \subseteq D$. To show that $D \subseteq ODCI_n\alpha ODCI_n$, let $im(\alpha) = \{a_1,a_2, . . . ,a_k\}$ and $\beta \in D$ be such that $rank(\beta) = m$, and $im(\beta) = \{b_1,b_2, . . . ,a_m\}$. Then $m \leq k$ and for each $i = \{1, . . . ,k\}$ we choose some element c_i in the set $A_i = \{x \in X_n : (x)\alpha = a_i\}$. Define λ, $\delta \in ODCI_n$ in the following

way: $dom(\lambda\} = dom(\beta\}$, $im(\delta\} = im(\beta\}$ *and for all* y *in* $B_j=\{z \in X_n : (z)\beta = b_j\}$, $j = 1, \dots, m$, *we set* $(y)\lambda = c_j$ *and* $(a_i)\delta = b_j$ $j = 1 \dots m$, *hence* $\lambda\alpha\delta = \beta$. *which implies* $\beta \in \mathrm{ODCI}_n\alpha\mathrm{ODCI}_n$, *thus,* $D \subseteq \mathrm{ODCI}_n\alpha\mathrm{ODCI}_n$, *and hence* $D = \mathrm{ODCI}_n\alpha\mathrm{ODCI}_n$.

Corollary 1 *All two-sided ideals of* $\mathrm{ODCI}_{[n]}$ *are principal and form the following chain:*

$$0 = I_0 \subset I_1 \subset \dots \subset I_{n-1} \subset I_n = \mathrm{ODCI}_{[n]} \quad (2)$$

we denote by

$$I_{(r)} = \{\alpha \in \mathcal{ODCI}_{[n]} : rank\,(\alpha) \le r\}, \qquad (0 \le r \le n) \quad (3)$$

a two-sided ideal of $\mathrm{ODCI}_{[n]}$, consisting of elements of rank not more than r.

Next we consider the following results.

Theorem 1 [[*18*],Theorem 6] *Let* $\alpha, \beta \in \mathrm{ODCI}_{[n]}$, *then*

1. $(\alpha, \beta) \in \mathcal{L}^* (\mathcal{ODCI}_{[n]})$ *if and only if* $im(\alpha) = im(\beta)$.
2. $(\alpha, \beta) \in \mathcal{R}^* (\mathcal{ODCI}_{[n]})$ *if and only if* $dom(\alpha) = dom(\beta)$.
3. $(\alpha, \beta) \in \mathcal{H}^* (\mathcal{ODCI}_{[n]})$ *if and only if* $\alpha = \beta$
4. $\mathcal{D}^* (\mathcal{ODCI}_{[n]}) = \mathcal{J}^* (\mathcal{ODCI}_{[n]})$

Recall that the relation K on the semigroup $\mathrm{ODCI}_{[n]}$ is defined as

$$\mathcal{K} = \mathcal{R}^* (\mathcal{ODCI}_{[n]}) \circ \mathcal{L}^* (\mathcal{ODCI}_{[n]}) \circ \mathcal{R}^* (\mathcal{ODCI}_{[n]})$$

where

$$\mathcal{K} = \mathcal{D}^* (\mathcal{ODCI}_{[n]}) = \mathcal{J}^* (\mathcal{ODCI}_{[n]})$$

The principal factor denoted I_r/I_{r-1} $(1 \le r \le n)$ or $K \cup \{0\}$ is Rees Quotent, in which the non-zero elements of I_r/I_{r-1} are of rank precisely r. The product of any two elements of I_r/I_{r-1} is 0 if it falls into the lower K-classes, otherwise is of rank r. Let

$$K_r = \{\alpha \in \mathcal{ODCI}_{[n]} : rank\,(\alpha) = r\} \quad (0 \le r \le n). \quad (4)$$

It is clear that I_r is the union of the sets $K_0, K_1, \dots, K_n$.

It is obvious that K_n contained exactly one element the identity $\{1_{[n]}\}$.

Now, we consider the elements in K_{n-1}-class as follows;

$$\varepsilon_1 = \begin{pmatrix} 2 & . & . & . & n \\ 2 & . & . & . & n \end{pmatrix} \tag{5}$$

$$\varepsilon_n = \begin{pmatrix} 1 & . & . & . & n-1 \\ 1 & . & . & . & n-1 \end{pmatrix} \tag{6}$$

$$\varepsilon_i = \begin{pmatrix} 1 & \dots & i-1 & i+1 & \dots & n-1 \\ 1 & \dots & i-1 & i+1 & \dots & n-1 \end{pmatrix}, (2 \leq i \leq n-1) \tag{7}$$

$$\beta_i = \begin{pmatrix} 1 & \dots & i-1 & i+1 & \dots & n \\ 1 & \dots & i-1 & i & \dots & n-1 \end{pmatrix}, (2 \leq i \leq n-1) \tag{8}$$

$$\gamma = \begin{pmatrix} 2 & 3 & . & . & . & n \\ 1 & 2 & . & . & . & n-1 \end{pmatrix} \tag{9}$$

Lemma 1 *There are n idempotents in* K_{n-1} *of* $\mathrm{ODCI}_{[n]}$.

Proof 4 *Each* K_r*-class consist of elements of rank r, therefore,* ${}^nC_r$ *gives the number of subsets of cardinality r which also gives the number of partial identities, hence* ${}^nC_{n-1}$ *gives the number of idempotents* K_{n-1}*-class.*

Lemma 2 *The set* K_{n-1} *contains exactly* $2n-1$ *elements.*

Proof 5 *From the above lemma,* K_{n-1}*-class consist of exactly n idempotents and exactly* 1 *isometry which is not an idempotent, any other order-preserving and orderdecreasing partial injective contractions can form in* $n-2$ *ways, thus we have* $n+1+$
$n-2=2n-1$.

An element $\alpha \in \mathrm{ODCI}_{[n]}$ is said to be undecomposable if it cannot be express as a product of elements say $\delta, \varphi \in \mathrm{ODCI}_{[n]} \setminus \{\alpha\}$ such that $\alpha = \delta\varphi$. The element $1_{[n]}$ is undecomposible in $\mathrm{ODCI}_{[n]}$. However, if $\alpha = \delta\varphi$ and $\alpha \in K_{n-1}$ then $\delta, \varphi \in K_{n-1}$ and $dom(\delta) = dom(\alpha)$ also, $im(\varphi) = im(\alpha)$.

Observe that $\varepsilon_i\beta_i = \beta_i$ and all other product of the elements in K_{n-1} are 0, as such, we deduce that;

Corollary 2 *The elements in* K_{n-1} *are undecomposible in* $\mathrm{ODCI}_{[n]}$.

Corollary 3 $K_{n-1} \subseteq \langle \gamma, \beta_i : i \in \{2, 3, \ldots, n-1\}, \varepsilon_i : i \in \{1, \ldots, n\} \rangle$.

Proposition 2 *For each* $0 \leq r \leq n-1$, $K_r \subseteq \langle K_{r+1} \rangle$

Proof 6 *Let* $\alpha \in K_r$ *be as in equation 1, we show that there exist* $\delta, \varphi \in K_{r+1}$ *such that* $\alpha = \delta\varphi$. *We consider cases for the elements in* $dom(\alpha) = \{a_1, a_2, \ldots, a_r\}$ *and* $im(\alpha) = \{b_1, b_2, \ldots, b_r\}$, *for* $a_i < a_{i+1}$, $b_i < b_{i+1}$ *and* $b_i \leq a_i$ *for all* i

case (1) if $| b_{i+1} - b_i | = | a_{i+1} - a_i |$ *all* $(i \in \{1, \ldots, r-1\})$,

subcase (1.1) $(a_1, b_1 > 1)$ *and* $(a_r, b_r = n)$ *then*

$$\alpha = \begin{pmatrix} a_1 - 1 & a_1 & . & . & . & a_r \\ b_1 - 1 & .b_1 & . & . & . & b_r \end{pmatrix} \begin{pmatrix} b_1 - 2 & .b_1 & . & . & . & b_r \\ b_1 - 2 & b_1 & . & . & . & b_r \end{pmatrix}$$

subcase (1.2) $(a_1, b_1 = 1)$ *and* $(a_r, b_r = n)$ *then*

$$\alpha = \begin{pmatrix} a_1 & a_1 + 1 & . & . & . & a_r \\ b_1 & .b_1 + 1 & . & . & . & b_r \end{pmatrix} \begin{pmatrix} b_1 & b_1 + 2 & . & . & . & b_r \\ b_1 & b_1 + 2 & . & . & . & b_r \end{pmatrix}$$

subcase (1.3) $(a_i, b_i \geq 2)$ *and* $(a_r, b_r = n)$ *then*

$$\alpha = \begin{pmatrix} a_1 & a_i + 1 & . & . & . & a_r \\ b_1 & .b_i + 1 & . & . & . & b_r \end{pmatrix} \begin{pmatrix} b_1 & b_i + 2 & . & . & . & b_r \\ b_1 & b_i + 2 & . & . & . & b_r \end{pmatrix}$$

subcase (1.4) $(a_1, b_1 = 1)$ *and* $(a_r, b_r < n)$ *then*

$$\alpha = \begin{pmatrix} a_1 & . & . & . & a_r & a_r + 1 \\ b_1 & . & . & . & b_r & b_r + 1 \end{pmatrix} \begin{pmatrix} b_1 & . & . & . & b_r & b_r + 2 \\ b_1 & . & . & . & b_r & b_r + 2 \end{pmatrix}$$

case (2) if $| b_{i+1} - b_i | = | a_{i+1} - a_i | = 2$ *for all* $(i \in \{1, \ldots, r-1\})$ *case(2.1)* $(a_1, b_1 = 1)$, $(a_r, b_r = n)$ *and there exist* $(j, l \in \{1, \ldots, r-1\})$ *such that*

$$\chi = \begin{pmatrix} a_1 & \ldots \ldots a_j\, b_j & a_j + 1\, b_j & a_{j+1} & \ldots a_l\, a_{l+1} & \ldots & a_r \\ b_1 & & + 1 & b_{j+1} & \ldots b_l\, b_{l+1} & \ldots & b_r \end{pmatrix}$$

$$\begin{pmatrix} b_1 & & b_{j+1} & \ldots b_l & b_l + 1 & b_{l+1} & b_r \\ b_1 & \ldots & b_j\, b_j\, b_{j+1} & \ldots b_l & b_l + 1 & b_{l+1} & \ldots \ldots b_r \end{pmatrix}$$

subcase (2.2) $(a_1, b_1 = 1)$, $(a_r, b_r = n)$ *and there exist* $(j, l \in \{1, \ldots, r-1\})$ *such* *that*

$$\alpha = \begin{pmatrix} a_1 & \dots & a_i & a_j+1 & a_{i+1} & \dots & a_r \\ b_1 & \dots & b_i & b_j+1 & b_{i+1} & \dots & b_r \end{pmatrix} \begin{pmatrix} b_1 & \dots & b_i & b_j+2 & b_{i+1} & \dots & b_r \\ b_1 & \dots & b_i & b_j+2 & b_{i+1} & \dots & b_r \end{pmatrix}$$

case (3) if $|b_{i+1} - b_i| = 1$, $|a_{i+1} - a_i| > 2$ *for some i. case (3.1)* $(a_1, b_1 = 1)$ *and* $(a_r = n)$

$$\alpha = \begin{pmatrix} a_1 & a_1+1 & . & . & . & a_r \\ b_1 & b_1+1 & . & . & . & b_r \end{pmatrix} \begin{pmatrix} b_1 & b_1+2 & . & . & . & b_r \\ b_1 & b_1+2 & . & . & . & b_r \end{pmatrix}$$

case (3.2) $(a_1, b_1 = 1)$ *and* $(a_r = n)$

$$\alpha = \begin{pmatrix} a_1 & . & . & . & a_{r-1}-2 & a_r \\ b_1 & . & . & . & b_{r-1}-2 & b_r \end{pmatrix} \begin{pmatrix} b_1 & . & . & . & b_{r-1}-1 & b_r \\ b_1 & . & . & . & b_{r-1}-1 & b_r \end{pmatrix}$$

case (3.3) $(a_1, b_1 = 1)$ *and* $(a_r = n)$ *there exist* $(j, l \in \{1, \dots, r-1\})$ *such that*

$$\alpha = \begin{pmatrix} a_1 & \dots \dots a_j & a_j+1 & a_{j+1} & \dots & a_l & \dots \dots & a_r \\ b_1 & \dots \dots b_j & b_j+1 & b_{j+1} & \dots & b_l & & b_r \end{pmatrix} \begin{pmatrix} b_1 & \dots \dots b_j & b_{j+1} & \dots & b_l & b_l+1 & b_{l+1} & \dots & b_r \\ b_1 & \dots \dots b_j & b_j. & \dots & b_l & +1 & b_{l+1} & \dots & b_r \end{pmatrix}$$

case (4) $(a_1, b_1 = 1)$, $(a_r = n, b_r = n-1)$ *there exist* $(j, l \in \{1, \dots, r-1\})$ *with*

case (4.1) $|b_{j+1} - b_j| = 1$, $|a_{l+1} - a_l| > 1$ *such that*

$$\alpha = \begin{pmatrix} a_1 & \dots & a_j & a_j+1 & a_{j+1} & \dots & a_r \\ b_1 & \dots & b_j & b_j+1 & b_{j+1} & \dots & b_r \end{pmatrix} \begin{pmatrix} b_1 & \dots & b_j & b_j+2 & b_{j+1} & \dots & b_r \\ b_1 & \dots & b_j & b_j+1 & b_{j+1} & \dots & b_{r-1} \end{pmatrix}$$

case (4.2) $|a_{j+1} - a_j| > 1$, $|b_{l+1} - b_l| > 1$ *such that*

$$\alpha = \begin{pmatrix} a_1 & \dots & a_j & a_j+1 & a_{j+1} & \dots & a_l & \dots\dots & a_r \\ b_1 & \dots & b_j & b_j+1 & b_{j+1} & \dots & b_l & \dots\dots & b_r \end{pmatrix}$$

$$\begin{pmatrix} b_1 & \dots & b_j & b_{j+1} & \dots & b_l & b_l+1 & b_{l+1} & \dots & b_r \\ b_1 & \dots & b_j & b_{j+1} & \dots & b_l & b_l+1 & b_{l+1} & \dots & b_{r-1} \end{pmatrix}$$

this completes the proof.

we now have the following

Corollary 4 *For* $0 \le r \le n-1$ $I_r = \langle K_r \rangle$.

Corollary 5 $\langle K_n \cup K_{n-1} \rangle = \mathrm{ODCI}_{[n]}$.

3 Maximal subsemigroups

We start with the following lemma which is also found in [4, 5]

Lemma 3 *Every maximal subsemigroup of* $\mathrm{ODCI}_{[n]}$ *contains the ideal* I_{r-2}.

Proof 7 *Let M be a maximal subsemigroup of* $\mathrm{ODCI}_{[n]}$. *if* $K_{n-1} \subseteq M$ *then by corollary 2,* $I_{r-2} \subseteq I_{r-1} = \langle K_{n-1} \rangle \subseteq M$. *If* $K_{n-1} \subsetneq M$, *then* $M \cup I_{r-2}$ *is a proper subsemigroup in* $\mathrm{ODCI}_{[n]}$ *and hence* $M \cup I_{r-2} = M$ *by maximality of M. This implies* $I_{r-2} \subset M$.

The main result on the maximal subsemigroups of the semigroup $\mathrm{ODCI}_{[n]}$.

Theorem 2 *A subsemigroup M of a semigroup* $\mathrm{ODCI}_{[n]}$ *is maximal if and only if is in one of the following four forms:*

(i) $M_{1_{[n]}} = \mathcal{ODCI}_{[n]} \setminus \{1_{[n]}\}$.

(ii) $M_\gamma = \mathrm{ODCI}_{[n]} \setminus \{\gamma\}$.

(iii) $M_{\epsilon_i} = \mathrm{ODCI}_{[n]} \setminus \{\epsilon_i\}$ $(i \in \{1,...,n\})$.

(iv) $M_{\beta_i} = \mathrm{ODCI}_{[n]} \setminus \{\beta_i\}$ $(i \in \{2,3,...,n-1\})$.

Proof 8 *Let* ϵ_i, β_i *and* γ *be defined as in equation 7, 8 and 9. It is clear that* $M_{1[n]}$ *is a maximal subsemigroup of* $\mathrm{ODCI}_{[n]}$,*since* $\mathcal{ODCI}_{[n]} \setminus \{1_{[n]}\} = I_{r-1}$ *and* $I_{r-1} \cup \{1_{[n]}\} = \mathcal{ODCI}_{[n]}$. *From corollary 2 & 3, we have the elements the* γ, $\beta_i\,(i \in \{2,3,...,n-1\})$ *and* $\epsilon_i\,(i \in \{1,...,n\})$ *are undecom-*

posible in K_{n-1} *and all the elements are needed to generate* K_{n-1}. *Therefore,* M_γ, $M_{\epsilon i}$ *and* $M_{\beta i}$ *are subsemigroups of* $\mathrm{ODCI}_{[n]}$. *It is clear that they are maximal since* $M_\gamma \cup \{\gamma\} = \mathrm{ODCI}_{[n]}$, $M_{\epsilon i} \cup \{\epsilon_i\} = \mathrm{ODCI}_{[n]}$ *and* $M_{\beta i} \cup \{\beta_i\} = \mathrm{ODCI}_{[n]}$.

Conversely, let M be a maximal subsemigroup of $\mathrm{ODCI}_{[n]}$. *Then* $M = I_{n-2} \cup B$, *where* $B \subset (K_{n-1} \cup K_n)$, *from lemma 3. If* $K_n \subsetneq B$ *then* $M \subseteq M_{1[n]}$ *since* $K_n = \{1_{[n]}\}$ *and thus* $M = M_{1[n]}$ *by the maximality of M. If* $K_n \subseteq B$, *then* $K_{n-1} \subsetneq B$ *since* K_{n-1}
$\subseteq \langle \gamma, \beta_i : i \in \{2, 3, \ldots, n-1\}\ \varepsilon_i : i \in \{1, \ldots, n\}\rangle$, *by corollary 3, the set B is contained in* $K_{n-1} \setminus \{\xi\}$ *for some* $\xi \in \{\gamma\ \beta_i\ \varepsilon_i : i \in \{2, 3, \ldots, n-1\}\}$. *Therefore,* $M = M_\gamma$, $M = M_{\epsilon i}$ *and* $M = M_{\beta i}$ *for* $i \in \{1, \ldots, n\}$, $i \in \{2, 3, \ldots, n-1\}$ *by the*
maximality M.

Corollary 6 *There are exactly* $2n - 2$ *maximal subsemigroups in* ODCI_n.

Proof 9 *The idempotents* ϵ_1, ϵ_n *can be express as product of an identity with another element, thus, we only consider the* $n - 2$ *idempotents, we can form order-preserving and order-decreasing elements in* $n - 1$ *ways and the identity element which give us*
$n - 2 + n - 1 + 1 = 2n - 2$.

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