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Application of One-Dimensional Heat Equation on the Temperature Distribution

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***Abstract**: Application of one-dimensional heat equation on the temperature distribution, the theory of Heat equation can be solved using an analytical system of separation of the variable of partial differential equation method to achieve a solution by using Fourier series which is the basis of Fourier transform, which of the integral transform in other to find the temperature distribution on a time. Based on the finding by solving using one-dimension on a rectangle surface in the temperature distribution on a surface of a dynamic system. Using Dirichlet, Neumann, and robin mixed conditions in cases 1 and 2 the solution is trivial and there is no temperature distribution in the system, while in case 3 the solution is not the trivial solution, in this case, the temperature distribution is flowing in rectangular coordinate system on a time.*

***Keywords**: Heat, Neumann, Dirichlet, Robin mixed, Fourier series and Dynamics*

1. Introduction

Variation of temperature over a period of time in a given region is known as heat equation for a function of (x,t), $(x,y,t)$and (x,y,z) of one two and three spatial variables, the time variable t. the equation is given by

$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$,where k is positive constant Abdulla (2018).

Heat equation and temperature are carefully connected, they are different, the Temperature has a magnitude and is a scalar quantity, whereas Heat transfer has direction as well as magnitude it is also a vector quantity. The driving force for any form of heat transfer is the temperature difference, in this case, the larger the temperature change, the larger the rate of heat transfer. However, the heat equation is a simple second-order partial differential equation that describes the variation of temperature distribution in a specified area over some time Yunus (2002) and Dabral. V *et al* (2011).

The implicit Euler system was used to solve parabolic and hyperbolic heat conduction problems and the results obtained show that the problems in the system are unconditionally stable Čiegis. R (2009).
The Two-dimensional heat equations can be written of the form $u_t = k\left(u_{xx} + u_{yy}\right)$

Related formulae hold:

$u = \iint G_2\left(x, y; x', y', t\right)\left(x', y'\right)dx'dy'$ with $G_2\left(x, y; x', y', t\right) = G_1\left(x, x'\right) = G_1\left(y, y', t\right)$ (ivrii, 2017).

The Robin boundary condition also identified as third kind, impedance, radiation, convective, partially absorbing, or reactive condition for a partial differential equation (PDE) enjoys a long history in the application of science and engineering (Sean, 2015).

A partial differential equation is commonly used to define laws in science and engineering, it is connected in physics, and it is suitable to use in the diffusion of heat flow, wave propagation, and Schrödinger equation in quantum mechanics. The Study of complex numerical systems for solving problems in science and engineering has mostly concentrated on methods for approximating models defined by partial differential equations (PDEs) (Bo Yang, 2019 and Erik B *et*al ,2015).

Analytical solutions of partial differential equations (PDEs) can be difficult or impossible in several cases, particularly when the domain of the problem is compound. Therefore, various numerical approaches were employed to solve the problems, numerical techniques for solving problems in science and engineering have mainly focused on methods for approximating models defined by partial differential equations (PDEs), while the significant coupling to the geometrical description of the domain has been mainly overlooked (Iman 2018 and Burman *et al* 2014).

Brociek. R (2016) study the numerical solution for space fractional heat conduction equation, Dirichlet and Robin boundary condition and Crank-Nicolson scheme was also used to illustrate the accuracy of described method and some computational examples will be presented as well. In all example the results are good, with an increase in the density of the grid and errors of estimated solution decreases was found. Author used fractional Riemann-Liouville derivative.

Johansson B. T eta (2011) present A very Good agreement through the analytical solution was also investigate; the technique of fundamental solutions show that the Dirichlet boundary conditions other than can be handled, Extensions to the three-dimensional time-dependent heat equation is suggested by transforming the fundamental solution.

Diagonal two –point block scheme formula of order four to solve directly the linear and nonlinear Robin boundary conditions, the approaches provides faster execution time, fewer number of total function calls in all verified problem and a good accuracy was achieved in solving the problems Nadirah, M .N *et al* 2018.

Finite volume formulation was established to use for two-dimensional models is extended to deal through axisymmetric models of heat conduction applications, the planned formulation is certified and shows to be real and flexible through the solution of simple model problems lyra P.R.M *et al* (2005).

Bo Yang (2019). Develop the two kind general Robin Boundary Value problems and generalize the robin condition for the scalar poison equation to the vector cases.

Karl (2018) Proposed a Random walk on the semi-cylinders process for solving interior and exterior mixed Dirichlet–Neumann–Robin boundary value problems for diffusion equations was proposed. The technique was the use of a broad class of domains of the semi-cylindrical type defined as a domain external to a set of parallel non-overlapping semi-cylinders of arbitrary cross-section, with an end caps match with the plane $\xi = 0$ On the end caps, a general Robin boundary condition is imposed, and the random walk is constructed so that it explicitly takings into account the boundary conditions, without simulation of the partial reflections. The system can be useful to solve different diffusion imaging problems. Applications to cathodoluminescence imaging and EBIC techniques are discussed where the approaches recommended are extremely well-organized.

Arturo (2007) present a second-order single-interpolation pattern that can be useful to enforce either Dirichlet, Neumann, or Robin conditions on the body surface was used to analyze heat transfer methods in the context of the Immersed Boundary Methods

The System of Monotone Operators and the Galerkin Technique In the study of a nonlinear solution One Dimensional Heat Equation through Mixed Boundary Conditions and the existence of globally well-defined strong solutions for this class of problems was achieved (Bonfim 2019).

The proposed two-point diagonally block method is suitable for solving the second-order Robin Boundary value problem BVPs straight with variable step-size strategy. the numerical outcome found was mange to reserve the exactness of the solution, on comparing the Result it was found that the result obtained was economically in terms of total steps and better in implementation time when compared with the existing technique (Nasir *et al* 2019).

Numerical solution of one-dimensional heat diffusion equation subject to Robin boundary conditions multiplied by a small parameter epsilon greater than zero. The result obtained reveals that the numerical solution of the differential equation by Robin boundary condition is exactly close in a certain sense of the analytic solution of the problem with homogeneous Dirichlet boundary conditions when ε tends to zero (LOZADA-CRUZ, *et al 2016)*.

A solution of one-dimensional Heat Equation by the method of separation of variables using FOSS tools Maxima was used. The results obtained by the separation of variables are the same as the results achieved by using the Maxima program (Sudha, *et al*, 2017).

Theobjectofthispaperistostudythebehaviourofthevariationalsolutionoftheequation $-\Delta u = f$ withmixedDirichlet-Robinboundaryconditions (Mghazli, 1992).

Sean D. L (2015) investigate the diffusion equation with a boundary condition that alternates between a Dirichlet and a Neumann condition at random to develop a Robin boundary condition and show that the mean of the solution satisfies a Robin condition in the limit of infinitely rapid switching rate with the proportion of time spent in the Dirichlet state.

2. METHODOLOGY

One dimensional Heat Equation

$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}\ u(x,t)$, $u(0,t) = u(l,t) = 0$ and $u(x,0) = f(x)$

Separation of variable

Let $u(x,t) = X(x)T(t)$ $\qquad u = XT$

$u(0,t) = 0 \Rightarrow X(0)T(0) = 0 \quad \forall t$ and $\therefore X(0) = 0$ $u(l,t) = 0 \Rightarrow X(l)T(t) = 0 \quad \forall t$ and $\therefore X(l) = 0$

$\frac{\partial u}{\partial t} = XT' \ \ and \ \ \frac{\partial u}{\partial x} = X'T,\ \frac{\partial^2 u}{\partial x^2} = X''T$ then $XT' = KX''T$

$\frac{T'}{kT} = \frac{X''}{X} = R$ Where R is constant $\Rightarrow X'' - Rx = 0$

Case 1. $R = \lambda^2 > 0$, and $\lambda > 0$ is positive

$X'' - \lambda^2 X = 0$ This is second order ordinary differential equation

$X(x) = Ae^{\lambda x} + Be^{-\lambda x}$ at $x = 0$ and $X(0) = 0 \Rightarrow A + B = 0$ $\therefore A = -B$

$X(l) = 0 \Rightarrow Ae^{\lambda l} + Be^{-\lambda l} = 0$ and $A(e^{2\lambda l} - 1) = 0, \quad \lambda > 0, l > 0$

If A = 0 B is also is zero B =0 which is trivial

Case 2 if $R = 0 \Rightarrow X'' = 0$ integrate twice then the solution becomes

$X = Ax + B$ if $X(0) = 0 \Rightarrow 0 = A + B$ since A = 0 and also B = 0, Again $X = Ax$

$X(l) = 0 \Rightarrow 0 = A.l + B \quad but \quad B = 0$

$\therefore Al = 0 \quad \Rightarrow A = 0$ Therefore solution is trivial

Case 3 $R = -\lambda^2 < 0, \quad \lambda > 0$

$m^2 + \lambda^2 = 0 \Rightarrow m = \pm i\lambda$

$X(x) = A\cos\lambda x + B\sin\lambda x$ and $x = 0 \Rightarrow X(0) = 0 \qquad A = 0$

$X(x) = B\sin\lambda x$ $X(l) = 0 \Rightarrow A\cos\lambda l + B\sin\lambda l = 0$ But $A = 0$

$X(x) = B\sin\lambda l = 0$

$\Rightarrow \sin\lambda l = 0$ For non-trivial solution for integer multiples of π which $\pi, 2\pi, 3\pi\cdots$

$$\therefore \lambda l = n\pi \quad \Rightarrow \lambda = \frac{n\pi}{l} \quad n = 1,2,3,\cdots$$

$$X(x) = B\sin\lambda x = B\sin\left(\frac{n\pi}{l}x\right)$$

$$X(x) = B_n \sin\left(\frac{n\pi}{l}x\right)$$

$$\frac{T'}{kT} = R \;\Rightarrow T' = kRT$$ if $R = -\lambda^2$

$$T' = -k\lambda^2 T$$

$$m = -k\left(\frac{n\pi}{l}\right)^2 = \frac{-kn^2\pi^2}{l^2}$$

$$T(t) = c_n e^{\frac{-kn^2\pi^2}{l^2}t} \quad n = 1,2,3,\cdots$$

$$u(x,t) = X(x)T(t)$$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) c_n e^{\frac{-kn^2\pi^2}{l^2}t}$$ Let $B_n c_n = A_n$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) e^{\frac{-kn^2\pi^2}{l^2}t}$$ Since $u(x,0) = f(x)$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) e^{\frac{-kn^2\pi^2}{l^2}t}$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l}x\right) e^{\frac{-kn^2\pi^2}{l^2}t}$$ since $A_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi}{l}x\right)dx$

3. Result

Problem 1. Solve the equations

$3u_t = u_{xx} \quad 0 \le x \le 2, \quad t > 0$

$u(0,t) = 0$ and $u(2,t) = 0$ $\qquad u(x,0) = x$

Separate the variable

$$3X(x)T'(t) = X''(x)T(t) \;\Rightarrow\; \frac{3T'}{T} = \frac{X''}{X} = \lambda \text{ where } \lambda \text{ is a constant and } X'' - X\lambda = 0$$

CASE 1: $R = \lambda^2 > 0$ $X'' - \lambda^2 X = 0 \;\Rightarrow X(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$ $X(0) = 0$ *and* $X(2)$ solution is trivial

CASE 2: R = 0 $X'' = 0 \;\Rightarrow c_1 x + c_2$ and $X(0) = 0$ *and* $X(2)$ the solution is trivial

CASE 3: $R = -\lambda^2 < 0$ $X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x \Rightarrow X(0) = 0$ *and* $c_1 = 0$ then $X(2) = 0$

$c_2 \sin 2\lambda = 0 \Rightarrow c_2 \sin 2\lambda = 0$ $\lambda_n = \frac{n\pi}{2} \Rightarrow X(t) = c_n \sin \frac{n\pi}{2}$

Then $3T' = T\lambda \;\Rightarrow T(t) = C_1 e^{\frac{1}{3}\lambda t}$

$$u(x,t) = X(t)T(t) = C_n \sin\left(\tfrac{n\pi}{2}\right) C_m e^{\frac{1}{3}\lambda t} \text{ and } f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\tfrac{n\pi}{2}\right) e^{\frac{1}{3}\lambda t}$$

By Fourier Series

$$b_n = \frac{2}{L}\int_0^2 f(x)\sin\left(\tfrac{n\pi x}{2}\right)dx = \int_0^2 x\sin\left(\tfrac{n\pi x}{2}\right)dx = \frac{-4}{n\pi}(-1)^{n-1}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{-4}{n\pi}(-1)^{n-1} \sin\left(\tfrac{n\pi}{2}\right) e^{\frac{-1}{3}\lambda t}$$

Problem 2

$$\frac{\partial u}{\partial t} = 5\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \; t > 0 \text{ and } \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial t}(\pi,t) = 0 \quad t > 0, \; u(x,0) = x \qquad 0 < x < \pi$$

$X'' - X\lambda = 0$, $T' - 3\lambda T = 0$ $X'(0) = X'(\pi) = 0$

Case 1 and 2 the solution is trivial

Case 3 $K = \mu^2 < 0, r^2 + \mu^2 = 0 \;\Rightarrow r = \pm i\mu$

$X(x) = c_1 \cos \mu x + \sin \mu x \quad \Rightarrow X'(x) = -\mu c_1 \sin \mu x + \mu c_2 \cos \mu x$

$X'(0) = 0 \;\Rightarrow c_2 = 0$ and $X'(\pi) = 0$

$-\mu c_1 \sin \mu\pi + \mu c_2 \cos \mu\pi = 0$ since c2 = 0, $\mu = n\pi$

$X_n = \cos n\pi x$

Then $T' = 5T\lambda \;\Rightarrow T(t) = C_1 e^{5\lambda t}$

$$u(x,t) = X(t)T(t) = C_n \cos(n\pi x) e^{5\lambda t} \text{ and } f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\tfrac{n\pi}{2}\right) e^{5\lambda t}$$

By Fourier Series $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx=\frac{2}{\pi}\int_0^{\pi}xdx=\pi$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nxdx=\frac{-4}{(2n-1)^2}\ for\quad n=odd$$

$$f(x)=\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{-4}{(2n-1)^2}\cos nx$$

Problem 3 solve $u_t=7u_{xx}\qquad 0<x<\pi,\quad t>0$

$u(0,t)=0\quad ,\quad u_x(\pi,t)=-u(\pi,t)\quad t>0\qquad u(x,0)=3$ the separate the variables

$$X(x)T'(t)=7X''(x)T(t)\qquad\qquad \Rightarrow\frac{T'}{7T}=\frac{X''}{X}=\lambda$$

$X''-\lambda X=0,\qquad T'-T\lambda=0$ case 1 and 2 the solution is trivial

Case 3: $\lambda=-\mu^2<0$ then $X(x)=c_1\cos\mu x+c_2\sin\mu x$

$X(0)=0\quad\Rightarrow c_1\cos 0+c_2\sin 0=0,\ then\ \ c_1=0$

$X'(x)=-c_1\mu\sin\mu x+c_2\cos\mu x\qquad X'(\pi)=c_2\mu\cos\mu\pi\ \ and\ \ -X(\pi)=-c_2\sin\mu\pi$

$c_2\mu\cos\mu\pi=-c_2\sin\mu\pi\qquad\qquad -\mu=\tan\mu\pi$

The above equation has infinite sequence of non-negative solution $0<\mu_1<\mu_2<\mu_3<\cdots$

For each n $\frac{2n-1}{2}<\mu<n\pi$ also as $n\to\infty,\ \ \mu_n=\frac{2n-1}{2}$ the temperature distribution tend to accelerate.

for $n>1$

$X_n=\sin nx\,,\quad T(t)=e^{-7\lambda t}\qquad\qquad u(x,t)=e^{-7\lambda t}\sin nx,$

$u(x,t)=\sum_{n=1}^{\infty}c_ne^{-7\lambda t}\sin nx$ as t = 0 then $u(x,0)=\sum_{n=1}^{\infty}c_n\sin nx$ by Fourier integral

$$c_n=\frac{2}{\pi}\int_0^{\pi}3\sin nxdx=\frac{6}{\pi n}\{1-\cos n\pi\}=\begin{cases}0 & if\ is\ even\\ \frac{12}{\pi n} & if\ \ n\ \ is\ \ odd\end{cases}$$

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