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Ideals and Green's Relations of the Semigroup of all Order-preserving and Order-decreasing Finite Partial Injective Contractions of a Finite Chain

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***Abstract**: Let $X_n = \{1,2, . . .,n\}$ be a finite chain and $ODCI_n$ be the semigroup of all order-preserving and order-decreasing partial injective contraction mappings. In this work characterisation of the ideals of the semigroup $ODCI_n$ is obtained analogous to the result obtained for the semigroups IO_n and CP_n, of all orderpreserving injections and of all partial contractions, also, $ODCI_n$ admit principal series. The Green's relations and its starred analogue is investigated for the semigroup $ODCI_n$ in contrast with the semigroups of all partial order-decreasing injections and OCI_n the semigroup all order-preserving partial injective contractions, $ODCI_n$ is ample.*

***Keywords:** Semigroup, Ideals, Order-decreasing, Partial Injective Contractions, Finite Chain, Principal Series, Ample Semigroup.*

1 Introduction and Preliminaries

Let $X_n =\{1,2, . . . ,n\}$ be a finite chain, a mapping α: $dom(\alpha) \subseteq X_n \rightarrow im(\alpha) \subseteq X_n$, where $dom(\alpha)$ and $im(\alpha)$ denote the domain and image of α. The mapping α is called a transformation, the set of all transformations on X_n under the operation of composition of mappings is associative and is called transformation semigroup. The transformation α is called partial if the domain is a subset of X_n, the set of all partial transformations is called partial transformation semigroup and denoted by P_n, α is called full or total transformation if the domain is equal to X_n, the set all full or total transformations is called full transformation semigroup and is denoted by T_n, the set of all partial injectives on X_n is denoted by I_n is also called the symmetric inverse semigroup.

Let α be an element in any of the semigroups P_n T_n or I_n, then α is said to be; orderpreserving (or order-reversing) if for all $x,y \in dom(\alpha)$ is such $x \leq y$ implies $x\alpha \leq y\alpha$ (respectively, $x\alpha \geq y\alpha$), order-preserving (or order-reversing) sometimes referred to isotone(or antitone) Dimitrova and Koppitz [15] ; is said to order-decreasing if for all $x,y \in dom(\alpha)$ $x\alpha \leq x$; is isometry or distance-preserving if $| x\alpha - y\alpha | = | x - y |$; is called a contraction if $| x\alpha - y\alpha | \leq | x-y |$ a contraction is sometimes called compression Zhao and Yang [1].

Let CP_n denote the semigroup of all partial contractions; PO_n denote the semigroup of all partial order-preserving transformations; OCP_n the semigroup of all partial order-preserving contractions; $ORCP_n$ be the semigroup of all order-preserving or order-reversing partial contractions; DP_n the semigroup of all partial isometries. Also, let CI_n, OCI_n, $ORCI_n$, $ODCI_n$ and DCI_n denote the semigroup of all partial injective contractions, the semigroup of all partial order-preserving injective contractions, the semigroup of all partial order-preserving or order-reversing injective contractions, the semigroup of all partial order-preserving and order-decreasing injective contractions and the semigroup of all partial order-decreasing injective contractions respectively, Umar and Al-kharousi [9].

For transformations $\alpha, \beta \in I_n$, we use the notation $\alpha\beta$ instead $\alpha \circ \beta$ and multiply from left to right using the left to right composition of transformations, that is, $x(\alpha\beta) = (x\alpha)\beta$. we define the cardinality $rank(\alpha) = |dom(\alpha)| = |im(\alpha)|$ since α is one-one and $rank(\alpha\beta) = min\{rank(\alpha), rank(\beta)\}$.

A non-empty subset A of a semigroup S is called a left ideal if $SA \subseteq A$, a right ideal if $AS \subseteq A$, and a two-sided ideal or an ideal if $SAS \subseteq A$. it is evident that every left, right and two-sided ideal is a subsemigroup. Among the ideals of a semigroup S, are S itself and if S contains zero element then $\{0\}$ is an ideal. if I is an ideal such that $\{0\} \subset I \subset S$ is called proper. A semigroup S is said to be simple if it cantains no proper ideal, a semigroup S containing zero is 0-simple if $\{0\}$ and S are the only ideals.

If a is any element of a semigroup S, the smallest left ideal of S containing a is $Sa \cup \{a\}$ and denoted by S^1a, which is the principal letft ideal generated by a. The principal right ideal generated by an element a is $aS \cup \{a\}$ and denoted by aS^1. The principal two-sided ideal or principal generated by an element a is $Sa \cup aS \cup SaS \cup \{a\}$ and denoted by S^1aS^1.

The study of ideals in semigroups results naturally in considering some equivalences, the following equavalences were introduced by Green's [29]. Let S be a semigroup, let $a, b \in S$, define an equivalence L on S by a L b if and only if $S^1a = S^1b$, that is, a and b generated the same principal left ideal; Similarly, a R b if and only if $aS^1 = bS^1$, that is, a and b generated the same principal right ideal,; a J b if and only if $S^1aS^1 = S^1bS^1$, that is, a and b generated two-sided principal ideal or an ideal; a H b if and only if a L b and a R b or H = L ∩ R; a D b if and only there exist $c \in S^1$ such that a L c and c R b, the equivalence D is also defined by D = L ∘ R, it is evident that L ∘ R = R ∘ L. If a is an element in a semigroup S, the equivalence classes the L, R, J , H and D-class containing the element a will be denoted by L_a, R_a, J_a, H_a and D_a respectively.

For starred Green's chatacteristions see; ([7],[8],[14],[26]), on a semigroup S the relation L^* is defined by the rule that $(a,b) \in L^*$ if and only if (a,b) are related by the Green's relation L in some oversemigroup of S. The relation R^* is defined dually. These relations also have the following characterisations:

$$\mathcal{L}^*(S) = \{(a,b) : for\ all\ x, y \in S^1, ax = ay \Longleftrightarrow xb = yb\}; \quad (1)$$

$$\mathcal{R}^*(S) = \{(a,b) : for\ all\ x, y \in S^1, xa = ya \Longleftrightarrow bx = by\}. \quad (2)$$

The join of the relations L^* and R^* is denoted by D^* and their intersection by H^*.

The Green's equivalences plays a fundamental role in developing the theory of semigroup, to understand the structure of any semigroup, the Green's characterisation is paramount. In an Attempt to develop the theory of semigroup, many reseachers studied various properties of transformation semigroups and in partcular the Green's equivalences on the semigroup P_n with some of its subsemigroups over the years, many interesting and delightful results were recorded. For example see [2],[3],[4],[5],[8],[12],[13], [22],[17],[6].

Umar [21] characterised Green's relations and their starred analogue for the semigroups, finite order-decrease full transformation semigroups and finite order-decrease partial one-one transformation semigroups. Ganyushkin and Mazorchuk [10] study the semigroup of all partial order-presrvering injections in which the ideals and Green's relations were characterised.

$$IO_n = \{\alpha \in I_n : (\forall\, x,y \in dom\alpha)\ x \le y \Longrightarrow x\alpha \le y\alpha\} \quad (3)$$

The algebraic study of CP_n, CT_n and CI_n the semigroup of Partial, full and partial oneone contractions was initiated by Umar and Al-kharousi in [9], in which the notations of the semigroups and its subsemigroups were given, as a result of these a number of literatures emerged concerning the semigroups of contractions. Zhoa and Yang [1] characterised the Green's relations and regularity of elements of the semigroup of partial order-preserving transformation and contractions

$$CPO_n = \{\alpha \in PO_n : (\forall\, x,y \in dom\ \alpha)\ |\ x\alpha - y\alpha\ | \le |\ x - y\ |\} = CP_n \cap PO_n \ (4)$$

Zubairu and Ali [30] characterised and obtained the number of the principal left (right) ideals of the semigroups CP_n and CT_n, also, computed the order of elements of rank(1) and rank(2). Ali *et al* [11] generalized the result of Zhao and Yang to the semigroup of partial contractions, in which among other results obtained the result of Zhao and Yang in

$$CP_n = \{\alpha \in P_n : (\forall\, x,y \in dom\alpha)\ |\ x\alpha - y\alpha\ | \le |\ x - y\ |\}. \quad (5)$$

2 Ideals in some semigroups

In this section we recall some results on ideals of some semigroups which are crucial to our investigation in some of the subsequent sections;

Lemma 1. [[*10*], proposition 1] *Let* $\alpha \in IO_n$, *then*

(1) the left principal ideal $IO_n\,\alpha$ equals $\{\beta: im(\beta) \subseteq im(\alpha)\}$

(2) the right principal ideal $\alpha\, IO_n$ equals $\{\beta: dom(\beta) \subseteq dom(\alpha)\}$

(3) the two-sided principal ideal $IO_n\,\alpha\, IO_n$ equals $\{\beta: rank\ (\beta) \le rank\ (\alpha)\}$

we now record another result from [30] we have theorem 1.3 as lemma 2(1) and theorem 2.1 as lemma 2(2) :

Lemma 2. [[*30*], Theorem 1.3 & 2.1]

(1) Let S denote the semigroup CP_n. For each $\alpha \in S$, the principal left ideal generated by α has the following form

$$S\alpha = \{\beta \in S: dom(\beta) \subseteq dom(\alpha)\ and\ \pi_\alpha \subseteq \pi_\beta\}.$$

(2) Let S denote the semigroup CP_n or CT_n. For each $\alpha \in S$, the principal right ideal generated by α has the following form $\alpha S = \{\beta \in S: im(\beta) \subseteq im(\alpha)\}$.

Lemma 3. [[6], proposition 4.1.2] *Each left* (*right or two − sided*) *ideal is a union of principal left* (*right or two − sided*) *ideals.*

Next, we employ an example to experiment the results considered in lemma 1 and lemma 2.

Example 1. *For any* $\alpha, \beta \in \{\mathcal{CP}_n, \mathcal{IO}_n\}$, *if* $\alpha = \begin{pmatrix} 2 & 3 & 5 & 7 & 8 \\ 3 & 4 & 6 & 7 & 8 \end{pmatrix}$ *and* $\beta = \begin{pmatrix} 4 & 6 & 8 & 10 \\ 1 & 2 & 3 & 5 \end{pmatrix}$ *then*

$$\alpha\beta = \begin{pmatrix} 2 & 3 & 5 & 7 & 8 \\ 3 & 4 & 6 & 7 & 8 \end{pmatrix} \begin{pmatrix} 4 & 6 & 8 & 10 \\ 1 & 2 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 5 & 8 \\ 1 & 2 & 3 \end{pmatrix}$$

Example 2. *For α, β as in example(1) above and any* $\lambda \in \{CP_n, IO_n\}$, *if*

$$\lambda = \begin{pmatrix} 1 & 4 & 5 & 7 & 8 & 9 \\ 1 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$$

then

$$\lambda\alpha\beta = \begin{pmatrix} 1 & 4 & 5 & 7 & 8 & 9 \\ 1 & 3 & 4 & 5 & 6 & 7 \end{pmatrix} \begin{pmatrix} 2 & 3 & 5 & 7 & 8 \\ 3 & 4 & 6 & 7 & 8 \end{pmatrix} \begin{pmatrix} 4 & 6 & 8 & 10 \\ 1 & 2 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix}$$

Remark 1. *We observed from lemma 1 , 2 , example 1 , 2 and ([30] lemma 1.1) the following:*

Lemma 4.

(i) *dom*(*αβ*) ⊆ *dom*(*α*)

(ii) *im*(*αβ*) ⊆ *im*(*β*)

(iii) *rank* (*αβ*) ≤ *min*{*rank* (*α*),*rank* (*β*)}

Remark 2.

(i) *For any α in* $\{CP_n, IO_n\}$, *the principal left ideals* $CP_n\alpha$ *or* $IO_n\alpha$ *of* CP_n *or* IO_n *respectively is determine by the image set of α.*

(ii) *For any α in* $\{CP_n, IO_n\}$, *the principal right ideals* αCP_n *or* αIO_n *of* CP_n *or* IO_n *respectively is determine by the domain set of α.*

For every *k*, 0 ≤ *k* ≤ *n*, denote $I_k = \{\beta \in IO_n : rank(\beta) \le k\}$.

Lemma 5. [[28], Proposition 2.3. [10], Corollary 1] *All two-sided ideals of* IO_n *are principal and form the following chain:*

$$0 = I_0 \subset I_1 \subset \ldots \subset I_{n-1} \subset I_n = IO_n \tag{6}$$

Recall that

Lemma 6. [[2] *proposition* 3 .1 .5] *If I, J are ideals of a semigroup S such that I ⊂ J and there is no ideal B of S such that I ⊂ B ⊂ J, then J/I is either 0-simple or null.*

Also, the Rees Quotient semigroup denoted by *J/I* or *J* ∪ {0} is either 0-simple or null. The semigroup *K*(*S*) and *J/I* are the principal factors, the product of two elements in *J/I* always falls into a lower J -class. If the factor is 0-simple then the product may lie in *J* or may fall into a lower J -class.

An ideal of a semigroup *S* is minimal if it does not properly contain any ideal. An ideal of a semigroup with zero is 0-minimal if the only proper ideal it contain is {0}. The unique minimal ideal is called a kernel and denoted *K* (*S*) . if *S* is a semigroup with zero, then *K* (*S*) = {0}.

A principal series of a semigroup *S* is a finite chain of ideals

$$K(S) = I_0 \subset I_1 \subset \ldots \subset I_{n-1} \subset I_n = S \quad (7)$$

that is maximal in the sense that there is no ideal *B* such that $I_i \subset B \subset I_{i+1}$. As such both IO_n admit pricipal series, however, not all semigroup admit pricipal series.

3 Order-preserving and order-decreasing partial injective contractions semigroup $ODCI_n$

Let α be an element in $ODCI_n$, we denote α in tabular form by;

$$\alpha = \begin{pmatrix} a_1 & a_2 & a_3 & . & . & . & a_r \\ \\ b_1 & b_2 & b_3 & . & . & . & b_r \end{pmatrix} (1 \leq r \leq n)$$

therefore, if α satisfies the following;

(i) For all $a_i, a_j \in dom(\alpha)$, $a_i \leq a_j$ implies $b_i \leq b_j$ ($i,j \in \{1,...,n\}$), thus α is order-preserving.

(ii) For all $a_i \in dom(\alpha)$, $b_i \leq a_i$ ($i \in \{1,...,n\}$), thus α is order-decreasing.

(iii) For all $a_i \in dom(\alpha)$, $| a_i\alpha - a_{i-1}\alpha | \leq | a_i - a_{i-1} |$ ($i \in \{1,...,n\}$), thus α is contraction.

(iv) For all $a_i, a_j \in dom(\alpha)$, $a_i \neq a_j \Longrightarrow a_i\alpha \neq a_j\alpha = b_i \neq b_j$, α is one-one.

then, α is order-presevering and order-decreasing partial injective contraction and denoted by

$$ODCI_n = \{\alpha \in OCI_n : \forall a_i, a_j \in dom(\alpha) \mid a_i\alpha - a_j\alpha \mid \leq \mid a_i - a_j \mid \textit{and } a_i\alpha \leq a_i\} \quad (8)$$

the semigroup of all order-presevering and order-decreasing partial injective contraction.

Remark 3. *It is clear that* $ODCI_n \subseteq IO_n \subseteq CP_n$. *As such, we will use some results concerning the two semigroups is our study.*

Next, from lemma 1 and 2 , we now have our main results on the ideals of $ODCI_n$.

Theorem 1. *Let* $\alpha \in ODCI_n$, *then the principal left ideal generated by* α *has the form*

$$ODCI_n\alpha = \{\beta \in ODCI_n : im(\beta) \subseteq im(\alpha)\} \quad (9)$$

Proof. Let $\alpha \in \mathrm{ODCI}_n$, suppose $A=\{\beta \in \mathrm{ODCI}_n: im(\beta) \subseteq im(\alpha)\}$. If $\delta \in \mathrm{ODCI}_n$, we have $(x)\delta\alpha = (x\delta)\alpha$. by lemma 4 $im(\delta\alpha) \subseteq im(\alpha)$ and thus $\mathrm{ODCI}_n\,\alpha \subseteq A$. Conversely, consider an arbitiry $\beta \in A$. We have $im(\beta) \subseteq im(\alpha)$. For each $b \in im(\beta)$ choose some a_b such that $(a_b)\alpha = b$. Consider a transformation δ for which $dom(\delta) = dom(\beta)$ and such that for $x \in dom(\beta)$ we have $x\delta = a_b$ and thus $(x)\delta\alpha = ((x)\delta)\alpha = (a_b)\alpha = b$, implies that $A \subseteq \mathrm{ODCI}_n\,\alpha$ as required. □

Theorem 2. *Let* $\alpha \in \mathrm{ODCI}_n$*, then the principal right ideal generated by* α *has the form*

$$\alpha\mathrm{ODCI}_n = \{\beta \in \mathrm{ODCI}_n: dom(\beta) \subseteq dom(\alpha)\} \tag{10}$$

Proof. Let $A = \{\beta \in \mathrm{ODCI}_n: dom(\beta) \subseteq dom(\alpha)\}$. Suppose $\delta \in \mathrm{ODCI}_n$ such that, $\alpha\delta = \beta$, we have $dom(\alpha\delta) = dom(\beta)$, but by lemma 4 $dom(\alpha\delta) \subseteq dom(\alpha) \subseteq dom(\beta)$, thus, $\mathrm{ODCI}_n \subseteq A$.

Conversely, suppose $\beta \in A$ and $dom(\beta) \subseteq dom(\alpha)$, implies that there exist δ *in* ODCI_n such that $\beta = \alpha\delta$ but $\alpha\delta \in \alpha\mathrm{ODCI}_n$. Hence $A \subseteq \mathrm{ODCI}_n$. □

Theorem 3. *Let* $\alpha \in \mathrm{ODCI}_n$*, then the principal two-sided ideal generated by* α *has the form*

$$\mathrm{ODCI}_n\alpha\mathrm{ODCI}_n = \{\beta \in \mathrm{ODCI}_n: rank\,(\beta) \leq rank\,(\alpha)\}. \tag{11}$$

Proof. Let $D = \{\beta \in \mathrm{ODCI}_n: rank\,(\beta) \leq rank\,(\alpha)\}$, by lemma 4 (iii) we have $\mathrm{ODCI}_n\alpha\mathrm{ODCI}_n \subseteq D$. To show that $D \subseteq \mathrm{ODCI}_n\alpha\mathrm{ODCI}_n$, let $im(\alpha) = \{a_1, a_2, \ldots, a_k\}$ and $\beta \in D$ be such that $rank\,(\beta) = m$, and $im(\beta) = \{b_1, b_2, \ldots, a_m\}$. Then $m \leq k$ and for each $i = \{1, \ldots, k\}$ we choose some element c_i in the set $A_i = \{x \in X_n: (x)\alpha = a_i\}$. Define $\lambda, \delta \in \mathrm{ODCI}_n$ in the following way: $dom(\lambda\} = dom(\beta\}$, $im(\delta\} = im(\beta\}$ and for all y in $B_j=\{z \in X_n: (z)\beta = b_j\}$, $j = 1, \ldots, m$, we set $(y)\lambda = c_j$ and $(a_i)\delta = b_j\, j = 1 \ldots m$, hence $\lambda\alpha\delta = \beta$. which implies $\beta \in \mathrm{ODCI}_n\alpha\mathrm{ODCI}_n$, thus, $D \subseteq \mathrm{ODCI}_n\alpha\mathrm{ODCI}_n$, and hence $D = \mathrm{ODCI}_n\alpha\mathrm{ODCI}_n$. □

Remark 4. *we have from lemma 5, 6 , remark 3 and equation 7 that the semigroup* ODCI_n *admit principal series, as such, obtained the immediate lemma*

Lemma 7. *All two-sided ideals of* ODCI_n *are principal and forms a chain*

$$0 = I_0 \subset I_1 \subset \ldots \subset I_{n-1} \subset I_n = \mathrm{ODCI}_n \tag{12}$$

Proof. Let I be a two-sided ideal in ODCI_n ,by the last statement of remark 2, for $k = max_{\beta\in I}\, rank\,(\beta)$, and $\alpha \in I$ be an element of $rank\,(k)$. Then by theorem 3 $I = \mathrm{ODCI}_n\alpha\mathrm{ODCI}_n = I_k$ □

An element a of a semigroup S is called regular if there exists $x \in S$ such that $axa = a$, if every element of S is regular then the semigroup S is said to be a regular semigroup. An element e of a semigroup S is called idempotent provided $e = e^2$, The set of all idempotent elements of semigroup S is denoted by E(S). An element a of a semigroup S with zero is called nilpotent provided that $a^k = 0$ for some $k \in \mathbb{N}$.

The semigroup ODCI_n consist of Idempotents which are partial identities, an idenpotent is a regular element, the set E(S) of all idempotents forms a subsemigroup which is a semilattice

4 Green's relations for the semigroup $ODCI_n$

Recall the Green's characterisation of the equivalences L, R, H, D, and J stated earlier. In this section we characterise the Green's relations for the semigroup $ODCI_n$, but first we consider some definitions and notations;

If U is a subsemigroup of a (not necessarily regular) semigroup S, if $a,b \in U$, there can be ambiguity about the meaning (for example) a L b, since L may stand for the appropriate Green equivalence either in S or in U. when confusion of this sort is likely to arise we shall distinguish between the two equivalences. Thus $(a,b) \in L(U)$ means that there exist $u, v \in U^1$ such that $a = ub$, $b = va$, while $(a,b) \in L(S)$ means that there exist $s, t \in U^1$ such that $a = sb$, $b = ta$. we shall use the notation $L(U) \subseteq L(S) \cap (U \times U)$. Similarly we can write the notations for the other equivalences.

The gap of the domain and image of a transformation α denoted $g\ (dom(\alpha))$ and $g\ (im(\alpha))$ is the ordered $(r-1)$ tuple, defined by

$$g\ (dom(\alpha)) = (a_2 - a_1, a_3 - a_2, \ldots, a_r - a_{r-1}) \text{ and } g\ (im(\alpha)) = (a_2\alpha - a_1\alpha, a_3\alpha - a_2\alpha, \ldots, a_r\alpha - a_{r-1}\alpha)$$

.

Next we have our main result of this section

Theorem 4. *Let* $\alpha, \beta \in ODCI_n$, *then*

(1) $(\alpha,\beta) \in L(ODCI_n)$ *if and only if* $im(\alpha) = im(\beta)$ *and* $g\ (dom(\alpha)) = g\ (dom(\beta))$.

(2) $(\alpha,\beta) \in R(ODCI_n)$ *if and only if* $dom(\alpha) = dom(\beta)$ *and* $g\ (im(\alpha)) = g\ (im(\beta))$.

(3) $(\alpha,\beta) \in H(ODCI_n)$ *if and only if* $\alpha = \beta$.

Proof. (1)

Let $(\alpha,\beta) \in L(ODCI_n)$, then

$\lambda\beta = \alpha$ and $\delta\alpha = \beta$ for some $\lambda, \delta \in ODCI_n$. Since $L(ODCI_n) \subseteq L(IO_n) \cap (ODCI_n \times ODCI_n)$, it follows that $im(\alpha) = im(\beta)$.

Conversely, let $dom(\alpha) = \{a_1, a_2, \ldots, a_r\}$ and $dom(\beta) = \{c_1, c_2, \ldots, c_r\}$, and suppose, $|\ a_i - a_j\ | = |\ c_i - c_j\ |$ for each $i, j \in X_n$, hence $g\ (dom(\alpha)) = g\ (dom(\beta))$.

□

Proof. (2)

Since $ODCI_n \subset IO_n$, and suppose $(\alpha,\beta) \in R(IO_n)$, so we have $(\alpha,\beta) \in R(ODCI_n)$ for each $a_i \in dom(\alpha)$, $c_i \in dom(\beta)$ implying $a_i = c_i$, then $dom(\alpha) = dom(\beta)$. Conversely, let $im(\alpha) = \{c_1, c_2, \ldots, c_r\}$ and $im(\beta) = \{d_1, d_2, \ldots, d_r\}$, suppose

$dom(\alpha) = dom(\beta)$ and $|\ c_i - c_j\ | = |\ d_i - d_j\ |$ for each $i, j \in X_n$, then we have $g\ (im(\alpha)) = g\ (im(\beta))$.

□

Proof. (3) Follows from proof (1) and (2). □

We now have the characterisation of the relations D and J

Theorem 5. *Let* $\alpha,\beta \in \mathrm{ODCI}_n$*. Then*

(1) $(\alpha,\beta) \in \mathcal{D}(\mathrm{ODCI}_n)$ *if and only if* $g(dom(\alpha)) = g(dom(\beta))$ *and* $g(im(\alpha)) = g(im(\beta))$

(2) $\mathcal{D}(\mathrm{ODCI}_n) = \mathcal{J}(\mathrm{ODCI}_n)$

Proof. (1)
Suppose that $(\alpha,\beta) \in \mathcal{D}(\mathrm{ODCI}_n)$. Then, there exists $\delta \in \mathrm{ODCI}_n$ such that $(\alpha,\delta) \in \mathcal{R}(\mathrm{ODCI}_n)$ and $(\delta,\beta) \in \mathcal{L}(\mathrm{ODCI}_n)$ by theorem 4. we have $dom(\alpha) = dom(\delta)$, $g(im(\alpha)) = g(im(\delta))$ and $im(\delta) = im(\beta)$, $g(dom(\delta)) = g(dom(\beta))$ which implies $g(dom(\alpha)) = g(dom(\beta))$ and $g(im(\alpha)) = g(im(\beta))$.
Conversely, suppose that $|\ im(\alpha)\ | = |\ im(\beta)\ |$,
$g(dom(\alpha)) = g(dom(\beta))$ and $g(im(\alpha)) = g(im(\beta))$ then by theorem 4. it follows that
$|\ a_{i+1}-a_i\ | = |\ c_{i+1}-c_i\ |$ and $|\ b_{i+1}-b_i\ | = |\ d_{i+1}-d_i\ |$ for each $(i \in \{1,2,...,r-1\})$, this completes the proof.

□

Proof. (2) The proof follows from the definition of $\mathcal{D}(\mathrm{ODCI}_n)$ and $\mathcal{J}(\mathrm{ODCI}_n)$ and the fact that ODCI_n is finite.

□

Lemma 8. *The semigroup* ODCI_n *contains regular elements. if fact all idempotents are regular*

Proof. □

Lemma 9. *For* $n = 1$ *the semigroups* IO_n *and* ODCI_n *coincide otherwise distinct and contains only the empty and identity maps, there is nothing to proof.*

Remark 5. *In view of* [[*14*], *corollary* 1.3] *and fact that the semigroup* ODCI_n *contain non-isometries for* $n \geq 2$ *we deduce the following lemmas*

Lemma 10. *For* $n \geq 2$ *the semigroup* ODCI_n *is irregular.*

A subsemigroup U of a semigroup S is called a full subsemigroup if it contains all the idempotents of S. It is called an inverse ideal of S if for all $u \in U$, there exists $u' \in S$ such that $uu'u = u$ and uu', $u'u \in U$.

Lemma 11. *The simegroup* ODCI_n *is an inverse ideal of* IO_n*.*

Proof. For each $\alpha \in \mathrm{ODCI}_n$, let $x \in dom(\alpha)$ and $y \in im(\alpha)$ be such that $x\alpha = y$. Then the mapping α' : $im(\alpha) \longrightarrow dom(\alpha)$ defined by $y\alpha' = x$ is in IO_n (in fact α' is the unique inverse of α in IO_n) and $\alpha\alpha'\alpha = \alpha$. Also, $\alpha\alpha' = 1_{dom(\alpha)}$ and $\alpha'\alpha = 1_{im(\alpha)}$. Thus, $\alpha\alpha'$, $\alpha'\alpha \in \mathrm{ODCI}_n$.

5 Starred Green's relations for the semigroup ODCI_n

Theorem 6. *Let* $\alpha, \beta \in \mathrm{ODCI}_n$*, then*

(1) $(\alpha,\beta) \in \mathcal{L}^*(\mathrm{ODCI}_n)$ *if and only if* $im(\alpha) = im(\beta)$.

(2) $(\alpha,\beta) \in \mathcal{R}^*(\mathrm{ODCI}_n)$ *if and only if* $dom(\alpha) = dom(\beta)$.

(3) $(\alpha,\beta) \in H^*(ODCI_n)$ *if and only if* $im(\alpha) = im(\beta)$ *and* $dom(\alpha) = dom(\beta)$ The proof is analogue of proof of Lemma 3.2.3 in [21].

Proof. (1) Certainly if $im(\alpha) = im(\beta)$ then $(\alpha,\beta) \in L(IO_n)$ and so $(\alpha,\beta) \in L^*(ODCI_n)$.

Conversely, if $(\alpha,\beta) \in L^*(ODCI_n)$ then by equation 1

$$\alpha\delta = \alpha\tau \Longleftrightarrow \beta\delta = \beta\tau \qquad \text{for all } \delta,\tau \in ODCI_n.$$

However, if we denote the partial injective identity map in $ODCI_n$ on a set A by id_A (A is any subset of X_n) then

$$x \notin im(\alpha) \Longleftrightarrow \alpha \cdot id_{\{x\}} = \alpha \cdot \phi$$

$$\text{i.e} \quad \Longleftrightarrow \beta \cdot id_{\{x\}} = \beta \cdot \phi \qquad \text{since } \alpha\, L^*\, \beta$$

$$\text{i.e} \quad \Longleftrightarrow x \notin im(\beta).$$

Thus, $im(\alpha) = im(\beta)$. □

Proof. (2) Certainly if $dom(\alpha) = dom(\beta)$ then $(\alpha,\beta) \in R(IO_n)$ and so $(\alpha,\beta) \in R^*(ODCI_n)$.

Conversely, if $(\alpha,\beta) \in R^*(ODCI_n)$ then by equation 2

$$\delta\alpha = \tau\alpha \Longleftrightarrow \delta\beta = \tau\beta \qquad \text{for all } \delta,\tau \in ODCI_n.$$

And

$$x \notin dom(\alpha) \Longleftrightarrow id_{\{x\}} \cdot \alpha = \phi \cdot \alpha$$

$$\text{i.e} \quad \Longleftrightarrow id_{\{x\}} \cdot \beta = \phi \cdot \beta \qquad \text{since } \alpha\, R^*\, \beta$$

$$\text{i.e} \quad \Longleftrightarrow x \notin dom(\beta).$$

Thus $dom(\alpha) = dom(\beta)$. □

to characterise the D^* as in [14], we consider the following lemma

Lemma 12. *For each* $\alpha \in ODCI_n$ *with* $|im(\alpha)| = p$, *there exists* $\beta \in ODCI_n$ *with* $im(\beta) = \{1,...,p\}$ *such that* $(\alpha,\beta) \in R^*(ODCI_n)$.

Proof. Let $dom(\alpha) = \{a_1,...,a_p\}$, be such that $a_i < a_{i+1}$ $(1 \le i \le p-1)$. Then $|a_{i+1} - a_i| \ge 1$ for each $i = 1,...,p-1$ and the mapping β defined by

$$a_i\beta = i \qquad (i = 1,...,p)$$

is in $ODCI_n$. Also, $im(\beta)=\{1,...,p\}$ and by theorem 6 (2) $(\alpha,\beta) \in R^*(ODCI_n)$.

□

On the semigroup $ODCI_n$ define the relation K by $(\alpha,\beta) \in K(ODCI_n)$
if and only if $|im(\alpha)| = |im(\beta)|$
by definition we have

$$D^*(ODCI_n) \subseteq K(ODCI_n). \tag{13}$$

now we have the following lemma

Lemma 13. *On the semigroup* $ODCI_n$,

$$K(ODCI_n) = R^*(ODCI_n) \circ L^*(ODCI_n) \circ R^*(ODCI_n)$$

Proof. Suppose that $(\alpha,\beta) \in K(ODCI_n)$, then $|im(\alpha)| = |im(\beta)| = p$ (say). Then by lemma 12, there exist $\delta, \gamma \in ODCI_n$ with $im(\delta) = im(\gamma) = \{1, \ldots, p\}$ such that $(\alpha,\delta) \in R^*(ODCI_n)$ and $(\gamma,\beta) \in R^*(ODCI_n)$ and, by theorem 6 (1), $im(\delta) = im(\gamma)$ implies $(\delta,\gamma) \in L^*(ODCI_n)$ and we have

$(\alpha,\beta) \in R^*(ODCI_n) \circ L^*(ODCI_n) \circ R^*(ODCI_n)$. Thus,

$$K(ODCI_n) \subseteq R^*(ODCI_n) \circ L^*(ODCI_n) \circ R^*(ODCI_n). \tag{14}$$

Conversely, suppose $(\alpha,\beta) \in R^*(ODCI_n) \circ L^*(ODCI_n) \circ R^*(ODCI_n)$. Then there exists $\delta, \gamma \in ODCI_n$ such that

$$(\alpha,\delta) \in R^*(ODCI_n),\ (\delta,\gamma) \in L^*(ODCI_n),\ (\gamma,\beta) \in R^*(ODCI_n).$$

Therefore,

$|im(\alpha)| = |im(\delta)|$, $|im(\delta)| = |im(\gamma)|$ and $|im(\gamma)| = |im(\beta)|$, and so $|im(\alpha)| = |im(\beta)|$. Thus $(\alpha,\beta) \in K(ODCI_n)$, that is

$$R^*(ODCI_n) \circ L^*(ODCI_n) \circ R^*(ODCI_n) \subseteq K(ODCI_n). \tag{15}$$

Therefore from equations 14 and 15 the result follows.

For $\alpha, \beta \in ODCI_n$, let $K(ODCI_n) = R^*(ODCI_n) \circ L^*(ODCI_n) \circ R^*(ODCI_n)$, then there exist $\delta, \gamma \in ODCI_n$ such that $(\alpha,\delta) \in R^*(ODCI_n)$, $(\delta,\gamma) \in L^*(ODCI_n)$ and $(\gamma,\beta) \in R^*(ODCI_n)$. Therefore we have $(\alpha,x_1) \in L^*(ODCI_n)$, $(x_1,x_2) \in R^*(ODCI_n)$, $(x_2,x_3) \in L^*(ODCI_n)$ and $(x_3,\beta) \in R^*(ODCI_n)$, were $x_1 = \alpha$, $x_2 = \delta$ and $x_3 = \gamma$. Hence by [[2] *proposition* 1.5.11] , $(\alpha,\beta) \in D^*(ODCI_n)$ and so $K(ODCI_n) \subseteq D^*(ODCI_n)$. (16)

Now, from equations 13 and 16, we have

$$K(ODCI_n) = D^*(ODCI_n) .$$

As in [14], we deduce that

Corollary 1. *Let* $\alpha,\beta \in ODCI_n$, *Then* $(\alpha,\beta) \in D^*(ODCI_n)$ *if and only if* $|im(\alpha)| = |im(\beta)|$.

Let L^* (R^*)-class containing the element *a* be denoted by L_a^* (R_a^*). We define a left (*right*)*-ideal of a semigroup *S* to be left (*right*)-ideal *I* of *S* for which $L_a^* \subseteq I (R_a^* \subseteq I)$ for all elements *a* of *I*. A subset *I* of *S* is a *-ideal if it is both a left *-ideal and a right *-ideal. The principal *-ideal $J^*(a)$ generated by the element *a* of *S* is the intersection of all *-ideals of *S* to which *a* belongs. The relation J^* is defined by the rule that: $(a,b) \in J^*$ if and only if $J^*(a) = J^*(b)$.

we now have the following lemma

Lemma 14. [[21],Lemma 1.2.2] *If a, b are elements of a semigroup S, then* $b \in J^*(a)$ *if and only if there are elements* $a_0, a_1, \ldots, a_n \in S$, $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\, S^1$ *such that* $a = a_0$, $b = b_n$ *and* $(a_i, x_i a_{i-1} y_i) \in D^*$ *for* $i = 1, \ldots, n$.

we now have the following lemma which is analogue of lemma 3.6 in [14]

Lemma 15. *For each* $\alpha, \beta \in ODCI_n$, $\alpha \in J^*\beta$ *implies* $|im(\alpha)| \leq |im(\beta)|$.

proof. Let $\alpha \in J^*\beta$, then by lemma 14 there exist $\beta_0, \ldots, \beta_n \in ODCI_n$, $\delta_1, \ldots, \delta_n, \gamma_1, \ldots, \gamma_n \in \mathcal{ODCI}_n^1$ such that $\beta = \beta_0$, $\alpha = \beta_n$ and $(\beta_i, \delta_i \beta_{i-1} \gamma_i) \in D^*(ODCI_n)$, for $i = 1, \ldots, n$. However, by corollary 1, this implies that $|im(\beta_i)| = |im(\beta_i \beta_{i-1} \gamma_i)| \leq |im(\beta_{i-1})|$ for all $i = 1, \ldots, n$, which implies $|im(\alpha)| = |im(\beta)|$ as required.

Next we consider some definitions which are important in the following results (see [26, 27, 21]).

A semigroup *S* in which each L^*-class and each R^*-class contains an idempotent is called abundant. An abundant semigroup in which the set of all idempotents E(*S*) is a semilattice is called adequate. For an element *a* of an adequate semigroup *S*, the (unique) idempotent in the L^*-class containing *a* will be denoted by a^* similarly, the (unique) idempotent in the R^*-class containing *a* will be denoted by a^+. An adequate semigroup *S* is said to *typeA* if $ea = a(ea)^*$ and $ae = a(ae)^+$ for all elements *a* in *S* and all idempotents *e* in *S*. A subsemigroup *U* containing all the idempotents of a semigroup *S* is called full subsemigroup.

Corollary 2. [[14],lemma 3.8] *The semigroup* $ODCI_n$ *is ample*

Proof. For each $\alpha \in ODCI_n$ then by definition $\alpha^* = 1_{im(\alpha)}$ and $\alpha^+ = 1_{dom(\alpha)}$. Let ϵ be an arbitrary idempotent in $ODCI_n$. Then obviously , $im(\epsilon\alpha) = im(1_{im(\epsilon\alpha)})$ and $dom(\epsilon\alpha) = dom(1_{dom(\epsilon\alpha)})$. Also

$$\epsilon\alpha = \alpha \cdot 1im(\epsilon\alpha) = \alpha(\epsilon\alpha)* \qquad \text{and} \qquad \alpha\epsilon = \alpha \cdot 1dom(\alpha\epsilon) = (\alpha\epsilon)+\alpha.$$

□

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