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Solution of One Dimensional Heat Equation Using Residue Theory

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Abstract: *The focused of this study is to show the solution of the one-dimensional heat equation using residue theory. One-dimensional heat equations with an initial and boundary value condition was solved by method of separation of variable of partial differential equation and transformed to second-order ordinary differential equation. However residue theory was employed to solve the homogenous differential equations and orthogonality of sine and cosine of Fourier series. In case one (1) and two (2) the result obtained using residue theory is the same as the other method of solving ordinary differential equation and both solutions are trivial. In case three (3) the root of the equation is complex therefore normal method of ordinary differential equations was apply to obtain the result. Residue theory is time-consuming than the other method of solving ordinary differential equation but is simpler, efficient and precise than the other methods. In short, the result of one-dimensional heat equation has numerous applications in physical problems and more often in science and technology innovations; hence the study recommended that the method should be applied into multidimensional heat transfer equation to solve scientific and engineering problems.*

Keywords: *heat equation, residue theory, separation of variable and boundary condition*

1. INTRODUCTION

One- dimensional initial value problem of the heat condition equation is given by

$$u_t = \alpha^2 u_{xx} \quad 0 < x < X,\ t > 0$$

With initial conditions of $u(x,0) = f(x) \quad 0 \le x \le X$ and the boundary condition $u(0,t) = h(t)$, $u(X,t) = g(t) \ t > 0$ and α is constant. Where f, h, and g are the prescribe functions of the variable. One-dimensional heat equations with initial boundary condition is considered for time-dependent, the problems of heat conduction is continuously being studied, and most of the problems are expressed with homogeneous boundary condition. Altiparmak (2003)

One- dimensional heat equation is of the form $\partial_t u(x,t) = \alpha \partial_{xx} u(x,t) \quad 0 \le x \le L, t > 0$ where $u(x,t)$ the independent variable and α is a constant coefficient. Peter (2016)

2. LITERATURE REVIEW

One dimensional heat equation was solved using the method of separation of variable known as Foss tools; boundary condition was given by the heat equations and algorithms for the maxima program was used to solve the heat equation. The result obtained from the analytical solution is the same with that of maxima program and there are numerous methods for the solution of one-dimensional heat equation apart from the Foss tools and maxima program (Sudha et al., 2017).

Ito's and Tanaka's types' formula related with X was determined to represent its solution of X as stochastic convolutions and linear stochastic heat equation with additive noise in one-dimensional heat equation was solved (Mihai et al., 2006).

A numerical solution of diffusion equation with restrictive pade approximations was studied. The study revealed that it has an advantage in the exact value at certain r, with high accuracy, and yield good results. According to the findings, comparative solutions with the other method have stayed made and the result is stable with the exact solution (Boz et al. 2016).

The finite difference method and finite element were used to solve the numerical solution of one-dimensional heat equation. The finite difference process does not converge frequently to an exact solution hence the amount of numerical instability was revealed. Finite element method was found to be the best method when compared with finite difference method to compute the numerical solutions (Ahmed et al., 2015).

Lattice Boltzmann method was used to solve radiative transfer equations and nonlinear energy equations. The effect of various parameters was also studied and the result found was compared to be consistent (Sagwook et al., 2009).

One-dimensional steady-state and transient phonon Boltzmann transport equation and COMSOL Multi-physics were successfully solved based on finite element and discrete ordinate methods for spatial and angular discretizations. The sensitivity study was conducted with various discretization refinements for different values of the Knudsen number, which is a measure of the Nano-scale regime. Sufficient refinement for angular discretization is critical in obtaining accurate solutions of the Boltzmann transport equation (Majchraz et al., 2014).

A numerical analysis of heating tissue by the two-temperature model and procedure of heating tissue is measured, which preserved as a porous medium and is separated into two part that its vascular "blood vessel" and extravascular "tissue", The heat conduction in the domain were measured and described by the two-temperature model which consisting of the system of two coupled equations. Assumption leads to the model created by the single partial differential equation were made. The stage of numerical computation has revealed that the assumed porosity of the blood and the tissue temperature differ slightly, and variant heating has a significant effect on the distribution of the temperature (Ronggui et al., 2005).

Simulation of Nano-scale Multidimensional Transient Heat Conduction Problems Using Ballistic-Diffusive Equations and Phonon Boltzmann Equation, different boundary conditions was employed to compare the simulation results with those obtained from the phonon BTE and the Fourier law, and the two-dimensional cases are simulated and the results is presented. The transient BTE is solved using the discrete ordinates method with a two Gauss-Legendre quadratures. Attention has been paid to the boundary conditions, and the result achieved from BDE is importantly improved than those from the Fourier law (Jordan, 2018).

Explicit Analytical solution of radiation diffusion equation by the double integrations technique of the integral-balanced have been creates, the strategy permits approximate closed-form solution to be created, step modification of the surface temperature and two problems time-dependent boundary condition have been solved, and error minimization of the approximate solution has been developed directly by limitations of the residual function of the governing equations (Suresh et al., 2018).

Analytical and numerical approaches were used in the correlation of the solution of two-dimensional steady-state heat conduction, the result obtained shows that the finite element are in good agreement with the analytical values and the solution reveals that it can be successfully used in complex thermal problems (Yang, 2016).

on the study of three dimensional heat equation, the solution shows that the process is highly consistent with the true fact and also the simulation is feasible and reliable by using MATLAB, it is reveal that the model is of universal value in solving the three dimensional heat equations. Hon et al (2005)

The numerical simulation analyses show that, the improved strategy of fourth-order convergence is effectively reduces the iterative number, and the results of numerical simulation experiments show that the inverse values match the exact values (Karamanli et al., 2012).

Symmetric smoothed particle hydrodynamics (SSPH) process was used to make the basic functions to solve 2D homogeneous and non-homogeneous steady-state heat transfer problems. Investigations are complete with the results obtained by using different weight functions and particle numbers. The error norms for three sample problems are computed by the use of two different kernel functions such as the revised Gauss function and revised super Gauss function, and the revised super Gauss function yields the smallest error norm. It is observed that the SSPH technique yields large errors for non-homogenous problems (Paulino et al., 2018).

Two applications were associated to prove the accurateness of the proposed formulation; it was observed that both spatial and time modifications were effective. The errors obtained in cylindrical and spherical coordinates were low and satisfactory, in both applications established [16].

Most of the cited literature they did not apply the method of Cauchy residue theory in the solution of one dimensional heat equation, in other to solve problems. But they applied various methods to solve it.

3. METHODOLOGY

One dimensional Heat Equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad u(x,t),\ u(0,t) = u(l,t) = 0 \text{ and } u(x,0) = f(x)$$

Separation of variable

Let $u(x,t) = X(x)T(t)$

$$u = XT$$

$$u(0,t) = 0 \Rightarrow X(0)T(0) = 0 \quad \forall t \text{ and } \therefore X(0) = 0$$

$$u(l,t) = 0 \Rightarrow X(l)T(t) = 0 \quad \forall t \text{ and } \therefore X(l) = 0$$

$$\frac{\partial u}{\partial t} = XT' \ \ and \quad \frac{\partial u}{\partial x} = X'T,\ \frac{\partial^2 u}{\partial x^2} = X''T$$

$XT' = KX''T$

$\frac{T'}{kT} = \frac{X''}{X} = R$ Where R is constant $\Rightarrow X'' - Rx = 0$

Case 1. $R = \lambda^2 > 0$, and $\lambda > 0$ is positive

$X'' - \lambda^2 X = 0$ This is second order ordinary differential equation

Cauchy's method for solving ordinary differential equations by residue theory was presented and to find the general solution for linear homogeneous differential equations with constant coefficients

Cauchy's technique for solving differential equations make use of buildup math was acquainted and with locate, the general solution for linear homogeneous differential equations with constant coefficients

$$a_0 y^n + a_1 y^{n-1} + \cdots + a_{n-1} y' + a_n y = 0 \tag{1}$$

Where $a_0 = 1 \ and \ a_j{}^{'s}, j = 1,2,3,\cdots$ are given constant s.

Theorem 1

Consider the differential equations with constant coefficients.

$$y^n + a_1 y^{n-1} + \cdots + a_{n-1} y' + a_n y = 0 \tag{2}$$

Let *f* be an arbitrary function of the complex variable *z*, whose zeros do not coincide with the zeros of the polynomial

$$g(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n \tag{3}$$

Then the general solution of (4.2.2) is given by

$$y(x) = \sum res\left[\frac{f(x)e^{zx}}{g(z)}\right] \tag{4}$$

We now show that (4) is a solution of the homogeneous differential equation (2). We assume that

$$y = \sum res\left[\frac{f(x)e^{zx}}{g(z)}\right]$$

Then

$$y' = \sum res\left[\frac{f(x)e^{zx}}{g(z)} . z\right]$$

$$y'' = \sum res\left[\frac{f(x)e^{zx}}{g(z)}.z^2\right]$$

$$\vdots$$

$$y^k = \sum res\left[\frac{f(x)e^{zx}}{g(z)}.z^k\right] \quad (k \ =1, 2,3\ldots n)$$

Hence

$$y^n + a_1 y^{n-1} + \cdots + a_{n-1}y' + a_n y \qquad \textbf{(5)}$$

$$= \sum \operatorname{Re} s\left[\frac{f(z)e^{zx}}{g(z)}z^n\right] + a_1 \sum \operatorname{Re} s\left[\frac{f(z)e^{zx}}{g(z)}z^{n-1}\right] + \cdots + a_{n-1}\sum \operatorname{Re} s\left[\frac{f(z)e^{zx}}{g(z)}z\right] + a_n \sum \operatorname{Re} s\left[\frac{f(z)e^{zx}}{g(z)}\right]$$

$$= \sum \operatorname{Re} s\left[\frac{f(z)}{g(z)}e^{zx}\left(z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n\right)\right] = \sum \operatorname{Re} s\left[\frac{f(z)}{g(z)}e^{zx}g(z)\right]$$

$$= \sum \operatorname{Re} s\left[f(z)e^{zx}\right] = 0$$

Since f(z) is analytic. Thus, (4) is indeed a solution of (2), i.e. (4) is a general solution

By the above theorem

$$X(x) = \sum res\left[\frac{f(z)}{g(z)}\right]$$

$$z^2 - \lambda^2 = 0 \quad \Rightarrow z = \lambda \; or \; z = -\lambda$$

$$X(x) = res\left(\frac{f(z)}{g(z)}e^{zx}, \lambda\right) + res\left(\frac{f(z)}{g(z)}e^{zx}, \lambda\right)$$

$$X(x) = \lim_{z\to\lambda}(z-\lambda)\frac{f(z)e^{zx}}{(z-\lambda)(z+\lambda)} + \lim_{z\to-\lambda}(z+\lambda)\frac{f(z)e^{zx}}{(z-\lambda)(z+\lambda)}$$

$X(x) = \dfrac{f(\lambda)e^{\lambda x}}{2\lambda} + \dfrac{f(-z)}{-2\lambda}e^{-\lambda x}$ Where $A = \dfrac{f(\lambda)}{2\lambda}$ $\;and\; B = \dfrac{f(-z)}{-2\lambda}$

$X(x) = Ae^{\lambda x} + Be^{-\lambda x}$ at $x = 0$ and $X(0) = 0 \quad \Rightarrow A + B = 0 \quad \therefore A = -B$

$X(l) = 0 \;\Rightarrow Ae^{\lambda l} + Be^{-\lambda l} = 0$ and $A\left(e^{2\lambda l} - 1\right) = 0, \quad \lambda > 0, l > 0$

If A = 0 B is also is zero B =0 which is trivial

Case 2 if $R = 0 \Rightarrow X'' = 0$ integrate twice then the solution becomes

$X = Ax + B$ if $X(0) = 0 \Rightarrow 0 = A + B$ since A = 0 and also B = 0, Again $X = Ax$

$X(l) = 0 \Rightarrow 0 = A.l + B \quad but \quad B = 0$
$\therefore Al = 0 \quad \Rightarrow A = 0$

Therefore solution is trivial

Case 3 $R = -\lambda^2 < 0, \quad \lambda > 0$

$m^2 + \lambda^2 = 0 \Rightarrow m = \pm i\lambda$ Apply Bromwich integral

$X(x) = A\cos\lambda x + B\sin\lambda x$ and $x = 0 \Rightarrow X(0) = 0 \quad A = 0$

$X(x) = B\sin\lambda x$

$X(l) = 0 \Rightarrow A\cos\lambda l + B\sin\lambda l = 0$ But $A = 0$

$X(x) = B\sin\lambda l = 0$

$\Rightarrow \sin\lambda l = 0$ For non-trivial solution for integer multiplies of π which $\pi, 2\pi, 3\pi \cdots$

$$\therefore \lambda l = n\pi \quad \Rightarrow \lambda = \frac{n\pi}{l} \quad n = 1,2,3,\cdots$$

$$X(x) = B\sin\lambda x = B\sin\left(\frac{n\pi}{l}x\right)$$

$$X(x) = B_n \sin\left(\frac{n\pi}{l}x\right)$$

$\frac{T'}{kT} = R \Rightarrow T' = kRT$ if $R = -\lambda^2$

$$T' = -k\lambda^2 T$$

$$m = -k\left(\frac{n\pi}{l}\right)^2 = \frac{-kn^2\pi^2}{l^2}$$

$$T(t) = c_n e^{\frac{-kn^2\pi^2}{l^2}t} \quad n = 1,2,3,\cdots$$

$$u(x,t) = X(x)T(t)$$

$$u(x,t)=\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) c_n e^{\frac{-kn^2\pi^2}{l^2}t} \quad \text{Let } B_n c_n = A_n$$

$$u(x,t)=\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) e^{\frac{-kn^2\pi^2}{l^2}t} \quad \text{Since } u(x,0)=f(x)$$

$$f(x)=\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) e^{\frac{-kn^2\pi^2}{l^2}t}$$

$$u(x,0)=f(x)=\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)$$

Note that

$$\int_0^l f(x)\sin\left(\frac{n\pi}{l}x\right)dx = \begin{cases} 0 & n\neq m \\ \frac{l}{2} & n=m \end{cases}$$

$$\int_0^l f(x)\sin\left(\frac{m\pi}{l}x\right)=\int_0^l \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{l}x\right)\sin\left(\frac{m\pi}{l}x\right)dx$$

$$f(x)\sin\left(\frac{m\pi}{l}x\right)dx=\sum_{n=1}^{\infty}\int_0^l A_n \sin\left(\frac{n\pi}{l}x\right)\sin\left(\frac{m\pi}{l}x\right)dx$$

$$if \quad n=m$$

$$\int_0^l f(x)\sin\left(\frac{n\pi}{l}x\right)dx = A_n\left(\frac{l}{2}+0+0+0+\cdots\right)$$

$$\int_0^l f(x)\sin\left(\frac{n\pi}{l}x\right)dx = \frac{A_n l}{2}$$

$$A_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi}{l}x\right)dx$$

$$u(x,t)=\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l}x\right) e^{\frac{-kn^2\pi^2}{l^2}t} \quad \text{since} \quad A_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi}{l}x\right)dx$$

4. RESULT

Problem1

$$4u_t = u_{xx} \qquad 0\leq x\leq 2,\ t>0$$

$u(0,t)=0$

$u(2,t)=0$

$u(x,0)=2\sin\left(\frac{\pi x}{2}\right)-\sin(\pi x)+4\sin(2\pi x)=f(x)$

Solution

Let $u(x,t)=X(x)T(t)$

$u=XT$

$u(2,t)=u(x,0)=0$

$4X(x)T'(t)=X''(x)T(t)$

$\frac{4T'(t)}{T(t)}=\frac{X''}{X}=R$

$\frac{4T'}{T}=R\Rightarrow \quad 4T'-RT=0$ and $\quad \frac{X''}{X}=R\Rightarrow \quad X''-RX=0$

Case 1 $R=\lambda^2>0, \qquad \lambda>0$ is positive

$X''-\lambda^2 X=0$

By residue theorem

$X(x)=res\left(\frac{f(z)}{g(z)}e^{zx},\lambda\right)+res\left(\frac{f(z)}{g(z)}e^{zx},-\lambda\right)$

$X(x)=\lim_{z\to\lambda}(z-\lambda)\frac{f(z)e^{zx}}{(z-\lambda)(z+\lambda)}+\lim_{z\to-\lambda}(z+\lambda)\frac{f(z)e^{zx}}{(z-\lambda)(z+\lambda)}$

$X(x)=\frac{f(\lambda)}{2\lambda}e^{\lambda x}+\frac{f(\lambda)}{-2\lambda}e^{-\lambda x}$

$X(x)=Ae^{\lambda x}+Be^{-\lambda x}$

$X(0)=0 \quad \Rightarrow A+B=0 \quad \Rightarrow A=-B$

$X(2)=0 \quad \Rightarrow Ae^{2\lambda}+Be^{-2\lambda}=0$ but A = -B

$\Rightarrow Ae^{2\lambda} - Ae^{-2\lambda} = 0 \quad \Rightarrow A\left(e^{2\lambda} - e^{-2\lambda}\right) = 0$

A = 0, B = 0 which trivial

Case 2. If R = 0

$X'' - RX = 0$

$X'' = 0$

$X(x) = Ax + B$

$X(0) = 0 \quad \Rightarrow B = 0 \qquad \therefore X(x) = Ax$

$X(2) = 0 \quad \Rightarrow 2A + B = 0 \quad but \quad B = 0,\ A = 0$ Therefore the solution is trivial

Case 3 if $R = -\lambda^2 < 0$

$X'' + \lambda^2 X = 0$ By ordinary differential Equation the solution will become
$X(x) = A\cos\lambda x + B\sin\lambda x$

$X(0) = 0 \qquad A + 0 = 0 \quad \Rightarrow A = 0 \qquad\qquad X(x) = B\sin\lambda x$

$X(2) = 0 \quad \Rightarrow A\cos 2\lambda + B\sin 2\lambda = 0$ But A=0 $\qquad B\sin 2\lambda = 0$

$\sin 2\lambda = 0$, $\lambda = \dfrac{n\pi}{2} \qquad n = 1,2,3,\cdots$

$$X_n(x) = \sin\left(\frac{nx\pi}{2}\right)$$

$$\frac{4T'}{T} = R \qquad T' = \frac{R}{4}T$$

$$T(t) = ce^{\frac{-\lambda^2}{4}t}$$

$$T_n(t) = c_n e^{\frac{-\left(n^2\pi^2\right)}{16}t} = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{2}\right) C_n e^{\frac{-n^2\pi^2 t}{16}}$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right) e^{\frac{-n^2\pi^2 t}{16}}$$

$$u(x,0) = f(x)$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right)$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$A_n = \int_0^2 \left(2\sin\left(\frac{\pi x}{2}\right) - \sin(\pi x) + 4\sin(2\pi x)\right)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$A_n = \int_0^2 2\sin\left(\frac{\pi x}{2}\right)\sin\left(\frac{n\pi x}{2}\right)dx - \int_0^2 \sin(\pi x)\sin\left(\frac{n\pi x}{2}\right)dx + \int_0^2 4\sin(2\pi x)\sin\left(\frac{n\pi x}{2}\right)dx$$

Which is Fourier sine series and we have use orthogonality of the series, that is

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0 & m \neq n \\ \frac{L}{2}, & m = n \end{cases}$$

Where L = 2, to solve for the individual $b_n = 0$ except for $n = 1,2,4$

$$b_1 = 2,\ b_2 = -1, b_4 = 4\ \ and\ b_n = 0\ \ \forall n$$

$$u(x,t) = 2\sin\left(\frac{\pi x}{2}\right)e^{\frac{-\pi^2 t}{16}} - \sin(\pi x)e^{\frac{-\pi^2 t}{4}} + 4\sin(2nx\pi)$$

Problem 2

$$u_t = \alpha^2 u_{xx} \qquad 0 \leq x \leq \pi, \quad t > 0$$

$$u(0,t) = 0$$

$$u_x(\pi,t) = 0$$

$$u(x,0) = 3\sin\left(\frac{5x}{2}\right) = f(x)$$

$$X(x)T'(t) = \alpha^2 X''(x)T(t)$$

$$\frac{X''}{X} = \frac{T'}{\alpha^2 T} = R$$

$X'' - XR = 0$ and $T' - \alpha^2 TR = 0$

Case 1 if $R = \lambda^2 > 0$, $\lambda > 0$ is positive

$X'' - \lambda^2 X = 0$ by residue theory

$X(x) = Ae^{\lambda x} + Be^{-\lambda x}$ and since $X(0) = 0 \Rightarrow A + B = 0 \therefore A = -B$

$X(\pi) = 0 \Rightarrow Ae^{\lambda\pi} + Be^{-\lambda\pi} = 0$

$\Rightarrow Ae^{\lambda\pi} - Ae^{-\lambda\pi} = 0$

$A\left(e^{\lambda\pi} - e^{-\lambda\pi}\right) = 0 \quad \lambda > 0, \quad \pi > 0$

Case 2 $R = 0 \Rightarrow X'' = 0$

$X(x) = Ax + B$ $\qquad X(0) = 0 \Rightarrow B = 0$ and $X(x) = Ax$

$X(\pi) = 0 \quad A\pi + B = 0$ But $B = 0$ and $A = 0$ therefore the solution is trivial

Case 3 $R = -\lambda^2 < 0, \quad \lambda > 0$

$X'' + \lambda^2 X = 0$

$X(x) = A\cos\lambda x + B\sin\lambda x$ $\qquad X'(x) = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$

$X(0) = 0 \Rightarrow A = 0$

$X(x) = B\sin\lambda x$

$X(\pi) = 0 \Rightarrow A\cos\lambda\pi + B\sin\lambda\pi = 0$

$X(\pi) = B\sin\lambda\pi$

$B\sin\lambda\pi = 0$

$$\lambda\pi = \frac{(2n-1)}{2}$$

$$\lambda = \frac{2n-1}{2} \qquad n = 1, 2, 3, \cdots$$

$$X_n(x) = B_n \sin\left(\frac{2n-1}{2}\right)x$$

$$\frac{T'}{\alpha^2 T} = R \quad \Rightarrow T' = R\alpha^2 T$$

$$T(t) = Ce^{R\alpha^2 t}$$

$$T_n(t) = C_n e - ^{\left(\frac{2n-1}{2}\right)\alpha^2 t}$$

$$u_n(x,t) = X_n T_n$$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\left(\frac{2n-1}{2}\right)x\right) C_n e - ^{\left(\frac{2n-1}{2}\right)\alpha^2 t}$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\left(\frac{2n-1}{2}\right)x\right) e - ^{\left(\frac{2n-1}{2}\right)\alpha^2 t}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\left(\frac{2n-1}{2}\right)x\right)dx$$

$$A_n = \frac{2}{\pi}\int_0^\pi 3\sin\left(\frac{5x}{2}\right)\sin\left(\left(\frac{2n-1}{2}\right)x\right)dx = \frac{3\pi}{2}$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\left(\frac{2n-1}{2}\right)x\right)$$

$$f(x) = \sum_{n=1}^{\infty} \frac{3\pi}{2}\sin\left(\left(\frac{2n-1}{2}\right)x\right)$$

$$3\sin\left(\frac{5x}{2}\right) = b_1 \sin\left(\frac{x}{2}\right) + b_2 \sin\left(\frac{3x}{2}\right) + b_3 \sin\left(\frac{5x}{2}\right) + \cdots$$

$$u(x,t) = 3\sin\left(\frac{5x}{2}\right)e^{-\left(\frac{5x}{2}\right)t}$$

Problem 3 $u_t = u_{xx} \quad 0 \le x \le 2\pi,\ t > 0$

$$u_x(0,t) = 0$$

$$u_x(2\pi,t) = 0$$

$$u(x,0) = x = f(x)$$

$\frac{T'}{T} = \frac{X''}{X} = R$

$X'' - RX = 0$ and $T' - RT = 0$

Case 1 $R = \lambda^2 > 0$

$X'' - \lambda^2 X = 0$ by residue theory we have

$X(x) = Ae^{\lambda x} + Be^{-\lambda x}$

$X'(x) = A\lambda e^{\lambda x} - B\lambda e^{-\lambda x}$

$X'(0) = 0 \quad \therefore A = B$ Also $X'(2\lambda) = 0 \quad A = B = 0$

Case 2 if $R = \lambda^2 = 0$

$X'' = 0$ Then $X(x) = Ax + B$ and $X'(x) = A$

$X'(0) = 0 \quad A = 0$ and $X'(2\pi) = 0 \quad A = 0$

Case 3 $R = -\lambda^2 < 0 \qquad \lambda > 0$

$X(x) = A\sin\lambda x + B\cos\lambda x$ and $X'(x) = A\sin\lambda x - B\sin\lambda x$

$X'(0) = 0 \quad A = 0$

$X(2\pi) = 0$

$A\sin\lambda 2\pi - B\sin\lambda 2\pi = 0$ but A =0

$-B\lambda\sin\lambda 2\pi = 0$ and $\lambda = \frac{n}{2} \quad n = 1,2,3,\cdots$

$X(x) = A\sin\lambda x + B\cos\lambda x$ we have $X(x) = B\cos\lambda x$

$X_n(x) = B\cos\left(\frac{n}{2}x\right)$

$\frac{T'}{T} = R$ therefore the solution will now become $T_n(t) = C_n e^{\frac{-n^2}{4}t}$

$u_n(x,t) = X_n(x)T_n(t) = \cos\left(\frac{n}{2}x\right)e^{\frac{-n^2 t}{4}}$

$$u(x,t)=\frac{A_0}{2}+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n}{2}x\right)e^{\frac{-n^2t}{4}}$$

$$u(x,0)=\frac{A_0}{2}+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n}{2}x\right)$$

Which we find the exploiting the orthogonality of cosine of cosines

$$\int_0^{2\pi}\cos\left(\frac{mx}{2}\right)\cos\left(\frac{nx}{2}\right)dx=\begin{cases}0 & m\neq n\\ \pi & m=n\neq 0\\ 2\pi & m=n=0\end{cases}$$

$$A_n=\frac{1}{\pi}\int_0^{2\pi}x\cos\left(\frac{nx}{2}\right)dx=\frac{1}{\pi}\left[\frac{2x}{n}\sin\left(\frac{n}{2}\right)\Big|_0^{2\pi}-\int_0^{2\pi}\frac{2}{n}\sin\left(\frac{nx}{2}\right)dx\right]$$

$$=\frac{1}{\pi}\left[\frac{2x}{n}\sin\left(\frac{nx}{2}\right)-\frac{2}{n}\left(\frac{-2}{n}\cos\left(\frac{nx}{2}\right)\right)\right]_0^{2\pi}=\frac{1}{\pi}\left[\frac{4}{n^2}\cos\left(\frac{nx}{2}\right)\right]_0^{2\pi}$$

$$A_n=\frac{4}{n^2\pi}\left[\cos\left(\frac{2\pi n}{2}\right)-1\right]=\frac{4}{n^2\pi}[\cos\pi n-1]$$

For n is even $A_n=0$, n is odd we have $A_n=\frac{-8}{n^2\pi}$ Therefore $A_{2m}=0$ and $A_{2m-1}=\frac{-8}{\pi(2m-1)^2}$

$$A_0=\frac{1}{2\pi}\int_0^{2\pi}xdx=\frac{1}{4\pi}\left[4\pi^2\right]=\pi$$

$$u(x,t)=\frac{A_0}{2}+\sum_{n=1}^{\infty}A_n\cos\left(\frac{nx}{2}\right)e^{-\left(\frac{n^2t}{4}\right)}=\frac{A_0}{2}+\sum_{m=1}^{\infty}A_{2m}\cos(mx)e^{-\left(m^2t\right)}+\sum_{m=1}^{\infty}A_{2m-1}\cos\left(\frac{(2m-1)x}{2}\right)e^{-\left(\frac{(2m-1)^2t}{4}\right)}$$

$$=\pi-\frac{8}{\pi}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\cos\left(\frac{(2m-1)x}{2}\right)e^{-\left(\frac{(2m-1)^2t}{4}\right)}$$

$$x=\pi-\frac{8}{\pi}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\cos\left(\frac{(2m-1)x}{2}\right)$$

5. Conclusion

One-dimensional heat equations with an initial and boundary value condition is solved by method of separation of variable of partial differential equation and transformed to second-order ordinary differential equation at time level and residue theory was employed to solve the homogenous differential equations and orthogonality of sine and cosine of Fourier series was used. In case one (1) and two (2) the result obtained using residue theory is the same as the other method and both solutions are trivial. In case three (3) the root of the equation is complex therefore normal method of ordinary differential equations was apply to obtain the result. Residue theory is time-consuming than the other method but is more simple, efficiency and precisely than the other method. In short, the result of one-dimensional heat equation has an application in many physical problems and come frequently in science, technology and innovation. It has been recommended that the method should be applied in multidimensional heat transfer equation to solve scientific and engineering problems.

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